
Physics 3

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Dimensional Analysis

1.1 A prediction before a detailed theory

Many physical questions begin before a complete model is available. Dimensional analysis uses a modest but powerful requirement: an equation must make sense in every consistent choice of units. It cannot ordinarily determine a pure number such as 2π , but it often finds the only possible combination of measurable quantities with the required units.

We begin with a classical oscillator. A small-angle pendulum has a length ℓ and moves under a gravitational field of magnitude g . Suppose that its period $\mathcal{T}$ depends only on these two quantities. Write the most general product of powers:

$$\mathcal{T} = C\ell^a g^b, \quad (1.1)$$

where C has no dimensions. Since

$$[\mathcal{T}] = T, \quad [\ell] = L, \quad [g] = LT^{-2},$$

matching the dimensions on both sides gives

$$T = L^a (LT^{-2})^b = L^{a+b} T^{-2b}.$$

Therefore $-2b = 1$ and $a + b = 0$, so $b = -1/2$ and $a = 1/2$. The period must scale as

$$\boxed{\mathcal{T} = C\sqrt{\frac{\ell}{g}}}. \quad (1.2)$$

For a real ideal pendulum at small amplitude, a dynamical calculation gives

$$\mathcal{T} = 2\pi\sqrt{\frac{\ell}{g}}. \quad (1.3)$$

The factor 2π contains no units, so dimensional analysis cannot discover it. Nor can Equation (1.2) reveal the amplitude correction that appears when the angle is not small. Those limitations are useful: they tell us exactly which information a scaling argument supplies and which information requires a model.

Dimensional analysis converts a physical claim into algebra on exponents. It identifies the allowed *scale* of an answer. Dimensionless constants and dimensionless parameters remain outside its reach.

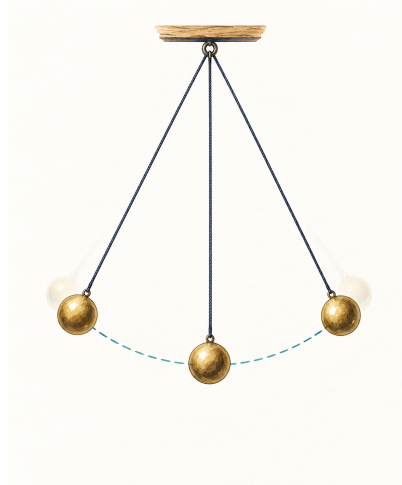


Figure 1.1: A simple pendulum in three positions. Dimensional analysis fixes the dependence $\mathcal{T} \propto \sqrt{\ell/g}$, but not its numerical coefficient.

1.2 Units and dimensions

A *unit* is a chosen standard, such as the metre or second. A *dimension* records the kind of physical quantity independently of the chosen unit. For the mechanics in this text, mass M , length L , and time T provide the central dimensions. Temperature Θ and amount of substance N enter several useful examples.

Table 1.1: Selected SI base quantities and common derived quantities.

Quantity	Symbol	SI unit	Dimensions
Length	x, ℓ, r	metre, m	L
Time	t	second, s	T
Mass	m	kilogram, kg	M
Temperature	T_{therm}	kelvin, K	Θ
Amount of substance	n	mole, mol	N
Velocity	v	m s^{-1}	LT^{-1}
Acceleration	a	m s^{-2}	LT^{-2}
Force	F	newton, N	MLT^{-2}
Energy or work	E, W	joule, J	ML^2T^{-2}
Power	P	watt, W	ML^2T^{-3}
Pressure	p	pascal, Pa	$ML^{-1}T^{-2}$

The familiar definitions provide quick checks. Force obeys $F = ma$, so

$$[F] = M(LT^{-2}) = MLT^{-2}.$$

Work equals force times displacement, $W = Fd$, and power measures energy transferred per time, $P = \Delta W/\Delta t$. Thus a watt has units

$$1 \text{ W} = 1 \text{ J s}^{-1} = 1 \text{ kg m}^2 \text{ s}^{-3}.$$

Every term joined by addition or subtraction must share the same dimensions. The equation

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

passes this test because each term has dimension L . The expression $x = x_0 + v_0 + a t^2$ does not: a length cannot be added to a velocity.

Numerical equality does not guarantee physical sense. For example, $5 \text{ m} + 5 \text{ s}$ has no meaning even though both numbers equal five. Always carry the dimensions as well as the numbers.

1.3 Sound in air at room temperature

The molar gas constant R , absolute temperature T_{therm} , and molar mass μ are natural choices for the sound speed in a dilute gas. Their dimensions are

$$[R] = ML^2 T^{-2} N^{-1} \Theta^{-1}, \quad [T_{\text{therm}}] = \Theta, \quad [\mu] = MN^{-1}.$$

The combination RT_{therm}/μ has dimensions $L^2 T^{-2}$. The only square root therefore has the dimensions of speed:

$$c_{\text{air}} = C \sqrt{\frac{RT_{\text{therm}}}{\mu}}. \quad (1.4)$$

Worked example: A room-temperature speed scale for sound

At $T_{\text{therm}} = 300 \text{ K}$, take $R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$ and $\mu = 0.029 \text{ kg mol}^{-1}$. Equation (1.4) with $C = 1$ gives

$$\sqrt{\frac{(8.31)(300)}{0.029}} \text{ m s}^{-1} \approx 2.9 \times 10^2 \text{ m s}^{-1}.$$

For an ideal gas, a fuller thermodynamic model supplies $C = \sqrt{\gamma}$, where $\gamma \approx 1.4$ for air. That predicts about $3.5 \times 10^2 \text{ m s}^{-1}$, close to the measured sound speed near room temperature. The missing factor is dimensionless, exactly as dimensional analysis permits.

1.4 A molecular length hidden in the properties of water

Measured bulk properties can reveal a microscopic scale. Consider water with density $\rho \approx 10^3 \text{ kg m}^{-3}$, dynamic viscosity $\eta \approx 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$, and sound speed $s \approx 1.5 \times 10^3 \text{ m s}^{-1}$. Seek a length

$$\ell_w = C \eta^a \rho^b s^c.$$

Since

$$[\eta] = ML^{-1} T^{-1}, \quad [\rho] = ML^{-3}, \quad [s] = LT^{-1},$$

requiring $[\ell_w] = L$ gives

$$a + b = 0, \quad -a - 3b + c = 1, \quad -a - c = 0.$$

The solution $a = 1$, $b = -1$, $c = -1$ leads to

$$\ell_w \sim \frac{\eta}{\rho s}. \quad (1.5)$$

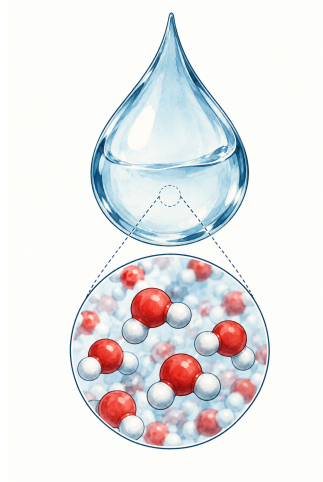


Figure 1.2: Bulk properties of water can reveal a molecular-length scale. The illustration links a macroscopic droplet to a magnified molecular view.

Worked example: Estimating a molecular scale of water

Substitution into Equation (1.5) gives

$$\ell_w \sim \frac{10^{-3}}{(10^3)(1.5 \times 10^3)} \text{ m} \approx 6.7 \times 10^{-10} \text{ m} = 0.67 \text{ nm}.$$

This is the scale of a few molecular diameters. A liquid's actual molecular spacing depends on structure and temperature, so the unknown dimensionless coefficient matters. The striking point is that macroscopic measurements alone lead to a nanometre-scale estimate.

1.5 Smallest length in the Universe: Planck scales

The water estimate belongs near atomic length scales. The Planck length is far smaller and arises from a different set of constants: the reduced Planck constant $\hbar$, the gravitational constant G , and the speed of light c . This gives

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35} \text{ m}. \quad (1.6)$$

Kinematics

Kinematics describes position, velocity, and acceleration without yet explaining their causes. Forces enter later. The essential language of kinematics is vector language, because where an object goes matters as much as how far it travels.

2.1 Vector review

2.1.1 Vectors and components

A scalar has magnitude only: mass, time, temperature, distance, and speed are scalars. A vector has magnitude and direction: displacement, velocity, and acceleration are vectors. In Cartesian coordinates, represent a vector by its component tuple:

$$\mathbf{a} = (a_x, a_y, a_z), \quad |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}. \quad (2.1)$$

For two-dimensional motion, simply omit the last component.

2.1.2 Addition and subtraction

Add vectors component by component:

$$\mathbf{a} + \mathbf{b} = (a_x + b_x, a_y + b_y, a_z + b_z). \quad (2.2)$$

Graphically, move the tail of $\mathbf{b}$ to the head of $\mathbf{a}$ without rotating either arrow. The resultant runs from the original tail to the final head. Subtraction means adding the opposite vector:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}). \quad (2.3)$$

The vector $-\mathbf{b}$ has the same magnitude as $\mathbf{b}$ and points in the opposite direction. This simple fact prevents a common mistake: vectors cannot usually be subtracted by subtracting their magnitudes.

2.1.3 The dot product

The dot product of two vectors is a scalar:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}| |\mathbf{b}| \cos \theta, \quad (2.4)$$

where θ is the angle between the vectors. It measures directional alignment. Parallel vectors have a positive maximum dot product, perpendicular vectors have dot product zero, and vectors that point generally opposite each other have a negative dot product.

If $\mathbf{u}$ is a unit vector, then $\mathbf{a} \cdot \mathbf{u}$ gives the signed component of $\mathbf{a}$ along $\mathbf{u}$. The vector projection is

$$\text{proj}_{\mathbf{u}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{u})\mathbf{u}. \quad (2.5)$$

Worked example: A displacement projected along a road

An object moves by $\mathbf{d} = (12, 5)$ m. A straight road points along the unit vector $\mathbf{u} = (4/5, 3/5)$. The signed displacement along the road is

$$\mathbf{d} \cdot \mathbf{u} = 12 \left(\frac{4}{5} \right) + 5 \left(\frac{3}{5} \right) = 12.6 \text{ m}.$$

The remaining component points across the road. The vector dot product therefore turns a two-dimensional displacement into a meaningful one-dimensional measurement.

2.2 Position, displacement, and path length

Choose an origin and coordinate axes. The position of a particle is the tuple

$$\mathbf{r}(t) = (x(t), y(t), z(t)). \quad (2.6)$$

Between initial time t_i and final time t_f , the displacement is

$$\Delta \mathbf{r} = (\Delta x, \Delta y, \Delta z) = (x_f - x_i, y_f - y_i, z_f - z_i), \quad \Delta t = t_f - t_i. \quad (2.7)$$

Displacement depends only on the endpoints. The distance d travelled is the length of the actual path, so it depends on the route.

The triangle inequality implies $|\Delta \mathbf{r}| \leq d$. Equality occurs only when the object follows a straight path without reversing direction. A round trip has zero total displacement but a positive distance.

2.3 Average velocity and average speed

Average velocity is a vector:

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{\Delta t}. \quad (2.8)$$

Average speed is a scalar:

$$v_{\text{avg}} = \frac{d}{\Delta t}. \quad (2.9)$$

They agree only in special cases, such as motion along a straight line with no reversal.

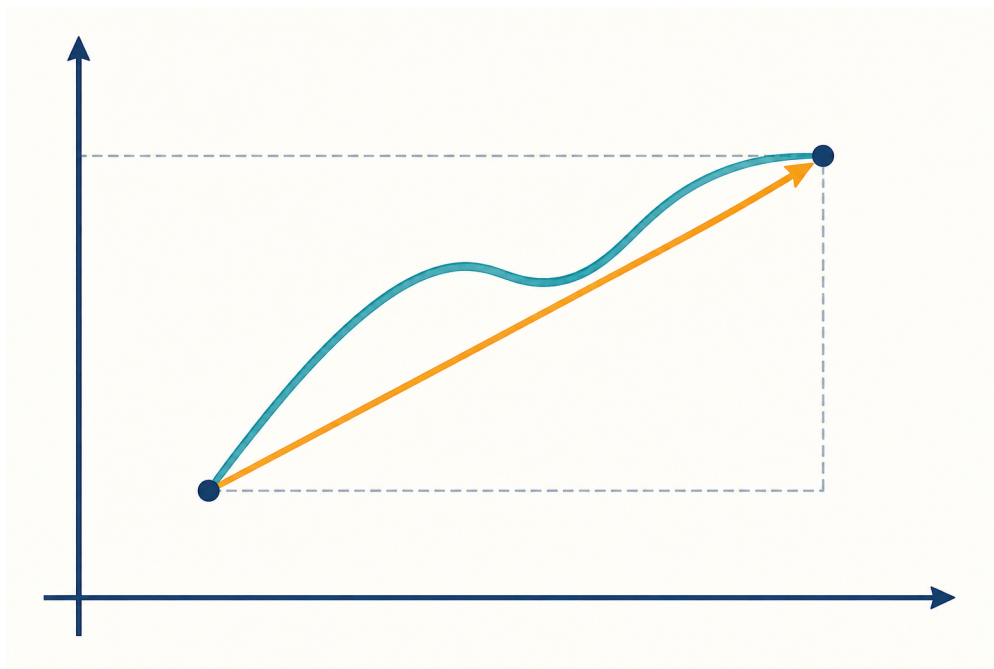


Figure 2.1: The curved trajectory determines the distance travelled. The displacement is the straight vector from initial to final position.

HOMework

- 1. Marine-animal power.** A marine animal moves through water at speed v . Its power-to-mass ratio P/M may depend on the water density ρ , the animal's size L , and its speed v :

$$\frac{P}{M} = C\rho^a L^b v^c,$$

where C is dimensionless.

- Use dimensional analysis to find a , b , and c .
 - At the same speed, does a smaller or larger animal require more power per unit mass? Briefly explain.
 - A giant squid has $L = 10$ m, $M = 200$ kg, and can reach $v = 10$ m s⁻¹. Taking $C = 1$, estimate its power during a rapid escape.
- 2.** A student walks from school to home at average speed u . The student then rides a bike back along the same route at average speed $3u$. If the one-way distance is D , find the average speed and the average velocity for the complete trip.
- 3.** The graph shows the position x of an object moving in one dimension. Find the average velocity and average speed over the interval $0 \leq t \leq 10$ s.

