

MATH 9: REVIEW OF LOGIC

October 25, 2026

1. LOGIC REVIEW

In logic, we work with **statements**: something that can be either true or false. Previously, we had discussed various logic operations:

- Logical AND: $P \wedge Q$ — true if both P, Q are true
- Logical OR: $P \vee Q$ — true if at least one of P, Q is true. Note that this includes the case when both P, Q are true.
- Negation: $\neg P$ — true if P is false, and false if P is true.
- Implication: $P \implies Q$. By definition, it is true in all cases except one: if P is true and Q is false.

These operations satisfy a number of logic laws, some of them listed below. All of them can be obtained from truth tables. The most important laws are

- **Modus Ponens**: given P and $P \implies Q$, we can conclude Q . In formulas, it can be written as

$$(P \wedge (P \implies Q)) \implies Q$$

- Transitivity of implication:

$$(P \implies Q \wedge Q \implies R) \implies (P \implies R)$$

- **Modus Nolens**: if $P \implies Q$, and we know that Q is false, then P is also false

$$((P \implies Q) \wedge \neg Q) \implies \neg P$$

- **De Morgan laws**:

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

2. QUANTIFIERS

We commonly use the following *quantifier statements*

- $\forall x : P(x)$ means *for all values of x , $P(x)$ is true*
- $\exists x : P(x)$ to mean *there exists a value of x such that $P(x)$ is true.*

Here $P(x)$ is statement which depends on some variable x .

In all of this, we assume that we know what set of values x can take (e.g., x is a real number). Sometimes we include the set of values in the quantifier: $\forall x \in \mathbb{Z} : x^2 > 0$.

In human language, we frequently also use other quantifier forms (some, not all, ...); all of them can be written using the the quantifiers above and basic logic operations.

The following property is used very frequently:

$$\neg(\forall x : P(x)) = \exists x : \neg(P(x))$$

$$\neg(\exists x : P(x)) = \forall x : \neg(P(x))$$

It is also common to use the shorthand notation $\nexists x : P(x)$ for $\neg(\exists x : P(x))$

3. PROOFS

Conditional proof: To prove $A \implies B$, assume that A is true and then deduce B .

Proof by contradiction: To prove some logical statement or theorem T , assume that T is false (or assume that the negation of T is true) and deduce a logical contradiction (e.g. a statement of the form $A \wedge \neg A$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

Proof of existence: To prove a statement of the form $\exists x : P(x)$, it is enough to find one example of x for which $P(x)$ holds true.

Proofs of universal quantifier: To prove a statement of the form $\forall x : P(x)$, you must provide a general argument which works for all possible values of x , that deduces that $P(x)$ is true.

Using existence quantifier A common way of reasoning is as follows. If we are given an existence statement $\exists x : P(x)$, we can say “let us choose a value x_0 such that $P(x_0)$ is true”. For example: if we are given that equation $x^3 + 3x - 17 = 0$ has a solution, we can say “let x_0 be a solution of this equation...”.

Warning: when doing it, you must choose a name for this value which has not been used before! You can not say “let z be a solution of...” if variable z was already used for something else before.

HOMEWORK

1. You are given the following statements:

$$A \wedge B \implies C$$

$$B \vee D$$

$$C \vee \neg D$$

Using this, deduce $A \implies C$.

2. Let $\mathbb{R}$ be the set of all integers. Prove $\forall x \in \mathbb{R} : x^2 + 2x + 4 > 0$.

You can use without proof some basic properties of arithmetic operations, such as associativity, commutativity, and distributivity; you can also use the fact that product of positive numbers is positive.

3. Let $\mathbb{Z}$ be the set of all integers. Are the following statements true or false? Can you prove them?

(a) $\forall n \in \mathbb{Z} : n^2 > 0$

(b) $\exists n \in \mathbb{Z} : n^2 > 0$

(c) $\forall n \in \mathbb{Z} : (n^3 > 0 \implies n > 0)$

4. Recall that an integer number n is called even if $\exists k \in \mathbb{Z} : n = 2k$. We will use notation $E(n)$ for statement n is even.

Prove the following:

(a) If n, m are even, then $m + n$ is even.

(b) If m is even and $m + n$ is even, then n is even

(c) If n is even, then for any m , mn is even.

(d) For any n , the number $n(n + 1)$ is even.

5. Write each of the following statements using only quantifiers, arithmetic operations, equalities and inequalities. In all problems, letters x, y, z stand for variables that takes real values, and letters $m, n, k, \dots$ stand for variables that take integer values.

(a) Equation $x^2 + x - 1$ has a solution

(b) Inequality $y^3 + 3y + 1 < 0$ has a solution

(c) Inequality $y^3 + 3y + 1 < 0$ has a positive real solution

(d) Number 100 is even.

(e) Number 100 is odd

(f) For any integer number, if it is even, then its square is also even.

6. For each of the statements of the previous problem, try to determine if it is true. If it is, give a proof. If not, disprove it (i.e., prove its negation). In both cases, you are only allowed to use basic properties of arithmetic operations, such as commutativity, associativity, etc.

7. (a) Write the statement “ n is the largest integer” using only quantifiers, arithmetic operations, equalities and inequalities.
- (b) Prove that the statement “1000 is the largest integer” is false
- (c) Prove that there doesn’t exist a largest integer, in other words: $\neg(\exists n : L(n))$, where $L(n)$ is the statement “ n is the largest integer”

*8. Consider the following arguments:

- (a) No homework is fun.
Some reading is homework.
Therefore, some reading is not fun.
- (b) All informative things are useful.
Some websites are not useful.
Some websites are not informative.

For each of them,

(1) write it in symbolic form, using quantifiers. Use set A for set of all human activities (which includes reading, writing, homework, etc) and notation $F(x)$ for “ x is fun”, $H(x)$ for “ x is homework”, etc.

(2) prove it, using the methods of proof discussed in class.

FYI: these are examples of two of Aristotle’s syllogisms, namely *Ferio* and *Baroco*. For more information, google Aristotle and syllogism.