

MATH 9
ASSIGNMENT 3: CENTER OF MASS
 OCT 4, 2026

CENTER OF MASS

Let $A_1, \dots, A_n$ be a collection of points and $m_1, \dots, m_n$ — some positive real numbers, representing masses placed at these points. Then the center of mass of such a collection of masses is defined to be a point M such that

$$\vec{OM} = \frac{m_1 \vec{OA_1} + \dots + m_n \vec{OA_n}}{m_1 + \dots + m_n}$$

where O is the origin.

One motivation for introducing this notion is the following. Suppose that we have a system of objects at positions $A_1, \dots$, with masses $m_1, \dots$, which are rigidly connected to each other. Then if we suspend the system at one point, it will be in equilibrium if and only if we place the suspension point at the center of mass M . More generally, for problems involving equilibrium, we can replace this system with a single object of mass $m = m_1 + \dots + m_n$ placed at the center of mass.

PROBLEMS

1. (a) Let M be a point on the segment A_1A_2 which divides this segment in proportion $MA_1 : MA_2 = 5 : 7$. Show that then, M is the same as the center of mass of the system consisting of mass $m_1 = 7$ at point A_1 and mass $m_2 = 5$ at point A_2 . [Hint: compare with problem 4 from the previous homework.]
 (b) Show that the center of mass of system of two points A_1, A_2 with masses m_1, m_2 is the point on the segment A_1A_2 , which divides this segment in proportion $MA_1 : MA_2 = m_2 : m_1$. In particular, if $m_1 = m_2$, then this point is the midpoint of A_1A_2 .
2. Show that if M is the center of mass of points $A_1, \dots, A_n$, then for any point X (not only for the origin), we have

$$\vec{XM} = \frac{m_1 \vec{XA_1} + \dots + m_n \vec{XA_n}}{m_1 + \dots + m_n}$$

(hint: $\vec{XM} = \vec{XO} + \vec{OM}$).

3. Show that the center of mass of some collection of points doesn't change if we replace two points A_1, A_2 with masses m_1, m_2 by a single mass $m_1 + m_2$ placed at the center of mass of A_1, A_2 .
4. Let M be the center of mass of a system of 3 points A, B, C with equal masses. Let A_1 be the midpoint of side BC , so that AA_1 the median of triangle $\triangle ABC$. Show that then M lies on the median AA_1 , dividing it in proportion $2 : 1$. Deduce from this that in fact, all three medians of a triangle pass through M (and thus intersect at a single point).
5. On the sides AB, BC, CD, DA of a parallelogram $ABCD$, mark points A_1, B_1, C_1, D_1 respectively, so that

$$AA_1 : A_1B = BB_1 : B_1C = CC_1 : C_1D = DD_1 : D_1A = 2 : 1.$$

Prove that the marked points $A_1B_1C_1D_1$ themselves form a parallelogram.

[Hint: denote $\vec{AB} = \vec{v}$, $\vec{AD} = \vec{w}$, and write vectors $\vec{AA_1}, \vec{AB_1}, \vec{A_1B_1}, \dots$ as combinations of $\vec{v}, \vec{w}$]

6. (a) In a triangle ABC , let point M_1 be on the side BC dividing it so that $M_1B : M_1C = 2 : 3$, and M_2, M_3 on sides AC, AB respectively so that

$$M_2A : M_2C = 2 : 5$$

$$M_3A : M_3B = 3 : 5$$

Prove that lines AM_1, BM_2, CM_3 intersect at a single point. [Hint: place appropriate masses at points A, B, C]

- (b) Prove Ceva's theorem: if points M_1, M_2, M_3 are on the sides BC, AC, AB respectively of triangle ABC , then lines AM_1, BM_2, CM_3 intersect at a single point if and only if

$$\frac{CM_1}{BM_1} \cdot \frac{BM_3}{AM_3} \cdot \frac{AM_2}{CM_2} = 1$$

[Hint: place appropriate masses at points A, B, C]