

MATH 9: ASSIGNMENT 2

SEPT 27, 2026

THALES THEOREM

Theorem 1. Let points A' , B' be on the sides of angle $\angle AOB$ as shown in the picture. Then lines AB and $A'B'$ are parallel if and only if

$$\frac{OA}{OB} = \frac{OA'}{OB'}$$

In this case, we also have $\frac{OA}{OB} = \frac{AA'}{BB'}$

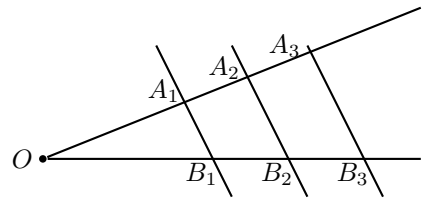
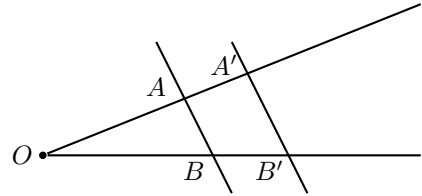
We have already seen and proved a special case of this theorem when discussing the midline of a triangle.

The proof of this theorem is unexpectedly hard. In the case when $\frac{OA}{OA'}$ is a rational number, one can use arguments similar to those we did when talking about midline. The case of irrational numbers is harder yet. We skip the proof for now; it will be discussed in Math 9.

As an immediate corollary of this theorem, we get the following result.

Theorem 2. Let points $A_1, \dots, A_n$ and $B_1, \dots, B_n$ on the sides of an angle be chosen so that $A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n$, and lines A_1B_1 , A_2B_2 , $\dots$ are parallel. Then $B_1B_2 = B_2B_3 = \dots = B_{n-1}B_n$.

Proof of this theorem is left to you as exercise.



SIMILAR TRIANGLES

Definition. Two triangles $\triangle ABC$, $\triangle A'B'C'$ are called *similar* if

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

and the corresponding sides are proportional, i.e.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

The common ratio $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ is sometimes called the similarity coefficient.

There are some similarity tests:

Theorem 3 (AAA similarity test). If the corresponding angles of triangles $\triangle ABC$, $\triangle A'B'C'$ are equal:

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

then the triangles are similar.

Theorem 4 (SSS similarity test). If the corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

then the triangles are similar.

Theorem 5 (SAS similarity test). If two pairs of corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

and $\angle A \cong \angle A'$ then the triangles are similar.

Proofs of all of these tests can be obtained from Thales theorem.

VECTORS

A **vector** is a directed segment. We denote the vector from A to B by $\overrightarrow{AB}$. We will also frequently use lower-case letters for vectors: $\vec{v}$.

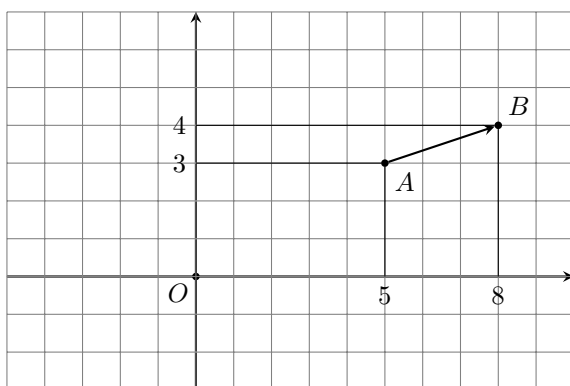
We will consider two vectors to be the same if they have the same length and direction; this happens exactly when these two vectors form two opposite sides of a parallelogram. Using this, we can write any vector $\vec{v}$ as a vector with tail at given point A . We will sometimes write $A + \vec{v}$ for the head of such a vector.

Vectors are used in many places. For example, many physical quantities (velocities, forces, etc) are naturally described by vectors.

VECTORS IN COORDINATES

Recall that every point in the plane can be described by a pair of numbers – its coordinates. Similarly, any vector can be described by two numbers, its x -coordinate and y -coordinate: for a vector $\overrightarrow{AB}$, with tail $A = (x_1, y_1)$ and head $B = (x_2, y_2)$, its coordinates are

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$



$$\overrightarrow{AB} = (8 - 5, 4 - 3) = (3, 1)$$

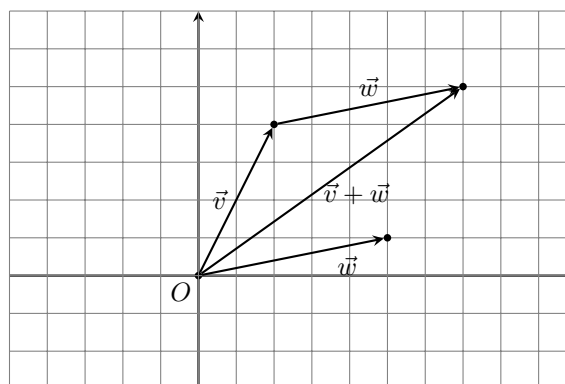
OPERATIONS WITH VECTORS

Let $\vec{v}, \vec{w}$ be two vectors. Then we define a new vector, $\vec{v} + \vec{w}$ as follows: choose A, B, C so that $\vec{v} = \overrightarrow{AB}$, $\vec{w} = \overrightarrow{BC}$; then define

$$\vec{v} + \vec{w} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

In coordinates, it looks very simple: if $\vec{v} = (v_x, v_y)$, $\vec{w} = (w_x, w_y)$, then

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y)$$



Theorem. *So defined addition is commutative and associative:*

$$\begin{aligned}\vec{v} + \vec{w} &= \vec{w} + \vec{v} \\ (\vec{v}_1 + \vec{v}_2) + \vec{v}_3 &= \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)\end{aligned}$$

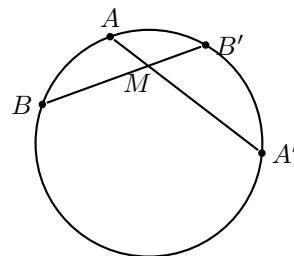
There is no obvious way of multiplying two vectors; however, one can multiply a vector by a number: if $\vec{v} = (v_x, v_y)$ and t is a real number, then we define

$$t\vec{v} = (tv_x, tv_y)$$

Again, we have the usual distributive properties: $t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$.

PROBLEMS

- Let M be a point inside a circle and let AA', BB' be two chords through M . Show that then $AM \cdot MA' = BM \cdot MB'$. [Hint: use inscribed angle theorem to show that triangles $\triangle AMB$, $\triangle B'MA'$ are similar.]



- Show that the angle bisector of a triangle divides the opposite side in the same proportion as the corresponding sides of the triangle: if CM is angle bisector of $\angle C$, then $\frac{AM}{BM} = \frac{AC}{BC}$. [Hint: drop perpendiculars from A, B to CM]
- Let $A = (x_1, y_1), B = (x_2, y_2)$. Show that the midpoint M of segment AB has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and that $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$.
[Hint: point M is $A + \frac{1}{2}\vec{v}$, where $\vec{v} = \vec{AB}$.]
- Let AB be a segment, and M — a point on the segment which divides it in the proportion 2:1, i.e., $|AM| = 2|MB|$. Let O be the origin. Show that $\vec{OM} = \vec{OA} + \frac{2}{3}\vec{AB} = \frac{1}{3}\vec{OA} + \frac{2}{3}\vec{OB}$
- Consider a parallelogram $ABCD$ and let $\vec{v} = \vec{AB}$, $\vec{w} = \vec{AD}$. Write the following vectors as combinations of $\vec{v}, \vec{w}$:
 - $\vec{AC}$
 - $\vec{BD}$
 - $\vec{AM}$, where M is the midpoint of BD
 Using this, can you prove that midpoint of BD coincides with the midpoint of AC ?
- Consider triangle ABC and let AA_1, BB_1, CC_1 be the medians of the triangle.
 - Write $\vec{AA_1}$ as combination of vectors $\vec{v} = \vec{OA}, \vec{w} = \vec{OB}, \vec{u} = \vec{OC}$.
 - Prove that then $\vec{AA_1} + \vec{BB_1} + \vec{CC_1} = 0$.
- In the notation of the previous problem, let M_1 be the point that divides median AA_1 in proportion 2 : 1 (i.e., $AM_1 : M_1A_1 = 2 : 1$). Write vector $\vec{OM_1}$ as a combination of $\vec{v}, \vec{w}, \vec{u}$. Repeat the same for all other medians.
 - Prove that the three medians of a triangle intersect at a single point.