MATH 8

COMBINATORICS REVIEW. LOGIC I: INTRODUCTION — SHORT 2026–27

This handout closes our combinatorics unit with a one-page summary and opens a new unit: **logic**. In combinatorics we *proved* a lot of things; now we step back and ask *what does it actually mean to prove something?*

Combinatorics Review

- **Addition and multiplication principles** (Handout 1): m ways *or* n ways (no overlap) gives $m+n$; m ways *followed by* n ways gives $m \cdot n$.
- **Permutations and combinations** (Handout 1):
  $P(n,k) = \dfrac{n!}{(n-k)!}$ (order matters), $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ (order doesn't matter).
- **Pascal's triangle** (Handout 2): $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. The same $\binom{n}{k}$ counts k-element subsets of an n-element set, grid paths with k steps right out of n, and binary strings of length n with k ones.
- **Binomial theorem** (Handout 3): $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n$.
- **Properties of Pascal's triangle** (Handouts 4–5): symmetry $\binom{n}{k} = \binom{n}{n-k}$; row sum 2^n; alternating row sum 0; hockey stick $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$; row n gives the digits of 11^n (with carrying); $\binom{n}{k}$ is odd exactly when k fits under n in binary.
- **Multinomials** (Handout 5): splitting n people into named groups of sizes $k_1, \ldots, k_r$ can be done in $\dfrac{n!}{k_1! \cdots k_r!}$ ways — the same number counts arrangements of `MISSISSIPPI`-type words.
- **Pigeonhole** (Handout 5): n objects in $k < n$ boxes put at least two objects in some box.
- **Stars and bars** (Handout 5): n identical objects into k bins: $\binom{n+k-1}{k-1}$ ways; $\binom{n-1}{k-1}$ if no bin is empty.
- **Methods:** *bijective proof* (match two collections one-to-one), *double counting* (count one collection two ways), and *complementary counting* (count the bad cases and subtract them from the total).

Propositions

A **proposition** is a statement that is either **true** (T) or **false** (F), but never both. "$2+2=4$" and "$5>10$" are propositions (true and false). "What time is it?" (a question), "Close the door" (a command), and "$x+3=7$" (its truth depends on x) are not. We name propositions with capital letters: A = "It is raining," B = "The ground is wet."

These true/false values are called **Boolean values**, after the self-taught English mathematician George Boole (1815–1864), who in 1854 turned logic into algebra. Eighty years later, a 21-year-old student named Claude Shannon noticed that Boole's algebra is exactly the language of electrical switches — and the digital age began.

NOT, AND, OR

A	B	$\neg A$	$A \wedge B$	$A \vee B$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

- **NOT** ($\neg A$) flips the truth value.
- **AND** ($A \wedge B$) is true only when *both* are true.
- **OR** ($A \vee B$) is true when *at least one* is true. In logic, "or" is always **inclusive**: "students who took calculus or physics may enroll" certainly lets in students who took both. (The everyday "soup or salad" kind of "or" is a different operation, XOR, coming next time.)

Truth Tables

A **truth table** lists the value of a compound proposition for every combination of values of its parts — with n variables, 2^n rows. Build it one column at a time:

A	B	C	$A \wedge B$	$(A \wedge B) \vee C$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Logic Laws

Two expressions are **logically equivalent** ($\equiv$) if they have the same truth value in every row. Many laws look like arithmetic, with AND as "multiplication" and OR as "addition":

- **Commutative:** $A \wedge B \equiv B \wedge A, \quad A \vee B \equiv B \vee A$.
- **Associative:** $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$, and the same for $\vee$.
- **Distributive:** $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ — like $a(b+c) = ab + ac$; but also $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$, which has no arithmetic twin!
- **Double negation:** $\neg(\neg A) \equiv A$.
- **Identity:** $A \wedge T \equiv A, \quad A \vee F \equiv A$. **Domination:** $A \vee T \equiv T, \quad A \wedge F \equiv F$.

De Morgan's Laws

$$\neg(A \wedge B) \equiv \neg A \vee \neg B, \qquad \neg(A \vee B) \equiv \neg A \wedge \neg B.$$

Negation flips AND $\leftrightarrow$ OR and pushes inside. "Alice is not (hungry and tired)" means "Alice is not hungry, *or* not tired"; "it is not (raining or snowing)" means "it is not raining *and* not snowing."

Common Mistakes

- **Misapplying De Morgan**: when you negate, change AND to OR (and vice versa) *and* negate each part.
- **Forgetting inclusive OR**: in logic, "A or B" includes the case where both are true.

Homework

1. How many different arrangements of the letters of `COMMITTEE` are there?

2. How many ways can you place 12 identical balls into 5 distinct boxes such that each box has at least 2 balls?

3. Which of these are propositions? For those that are, say whether they are true or false.

   (a) "100 is a perfect square."
   (b) "How old are you?"
   (c) "$2^{10} > 1000$."
   (d) "n is odd."
   (e) "There are more than a million prime numbers."
   (f) "Stop talking!"

4. Let R be "it is raining," W be "it is windy," and C be "it is cold." Write in symbols:

   (a) It is raining and windy, but not cold.
   (b) It is neither raining nor windy.
   (c) It is cold, or it is raining and windy.
   (d) It is not both raining and cold.

5. Write truth tables for:

   (a) $\neg(A \wedge B)$
   (b) $(A \vee B) \wedge \neg A$
   (c) $\neg A \wedge (B \vee C)$

6. Simplify using logic laws:

   (a) $\neg(\neg P \wedge \neg Q)$
   (b) $A \wedge (A \vee B)$
   (c) $(A \vee B) \wedge (\neg A \vee B)$

7. Find all truth values of A, B, C that make $(A \vee B) \wedge (\neg A \vee C) \wedge \neg C$ true.

8. Write the negation of each statement in simple English:

   (a) "The test is hard and the time is short."
   (b) "I will study math or physics tonight."
   (c) "She is smart and hardworking."

9. Simplify $\neg(A \vee \neg B) \vee (A \wedge B)$, naming the law you use at each step.

10. **Restaurant menu.** A restaurant menu says: *"The fixed-price dinner includes an entree, a dessert, and soup or salad."* Using the basic propositions

    - E: your dinner includes an entree
    - D: your dinner includes a dessert
    - P: your dinner includes a soup

- S: your dinner includes a salad

write this menu policy as a logical formula. (Be careful: the "or" here is inclusive — some restaurants let you get both, others won't — but in standard English menu phrasing it's read as "at least one of soup or salad.")

11. A door lock opens when you enter the correct code *and* either swipe your card or scan your fingerprint. Let K be "correct code entered," D be "card swiped," and F be "fingerprint scanned."

    (a) Write a formula for "the lock opens."

    (b) Write a formula for "the lock stays shut," and use De Morgan's laws to push the negation all the way inside. Say the result in plain English.

12. H **Knights and Knaves:** On the island of knights and knaves, knights always tell the truth and knaves always lie. You meet two people, Bart and Ted. Bart claims: "I and Ted are both knights or both knaves." Ted tells you: "Bart would tell you that I am a knave." Who is a knight and who is a knave?

13. M **Knights and Knaves.** Three islanders A, B, C. A says: "B and C are both knights." B says: "A is a knave or C is a knave." C says: "B is a knave." Who is a knight and who is a knave?

14. M **Majority vote.** Three judges A, B, C vote yes (T) or no (F), and a proposal passes when at least two of them vote yes.

    (a) Write a formula using $\wedge$, $\vee$ that is true exactly when the proposal passes.

    (b) Show that $(A \vee B) \wedge (A \vee C) \wedge (B \vee C)$ is another answer.