MATH 8

COMBINATORICS REVIEW. LOGIC I: INTRODUCTION — FULL 2026–27

Introduction

This handout has two parts. The first closes our combinatorics unit: a compact *reference summary* of the major formulas and ideas from Handouts 1–5, organized by topic, together with a few mixed review problems. Treat it as the place to come back to whenever you've forgotten which formula goes with which problem. The summary does not re-prove anything — each item points to where the full discussion lives. The real take-away from the unit is the habit of *recognizing which technique a problem wants*, and the review problems are there to practice exactly that.

The second, larger part opens a new unit: **logic**.

Combinatorics Review: Summary of Key Concepts

Historical Note: Combinatorics, Then and Now

Counting problems are among the oldest in mathematics. Sanskrit poets in ancient India enumerated rhythmic patterns; Chinese commentators on the *I Ching* counted hexagram arrangements; Hellenistic Greeks computed how many words could be formed from a set of letters. The subject took its modern shape in 1654: Pascal and Fermat exchanged letters on a gambling puzzle and founded probability theory, and in the same year Pascal wrote his *Treatise on the Arithmetical Triangle* — the triangle you now know so well.

For the next two centuries, combinatorics grew quietly alongside probability and number theory — Bernoulli, Euler, Montmort, Lagrange, Gauss, and Dirichlet all contributed pieces. But its modern explosion came in the 20th century, when entirely new applications appeared: *Frank Ramsey* (1930) showed that complete disorder is impossible (Ramsey theory); *Claude Shannon* (1948) used combinatorial arguments to invent **information theory** and so create the mathematical foundation of digital communication; *Paul Erdős* (1913–1996) spent his life launching whole subfields and posing problems by the thousand. Today, combinatorics is the lingua franca of computer science: every running-time analysis, every cryptographic protocol, every search algorithm rests on counting arguments of the kind you have been learning.

The five handouts you've just finished give you the foundational vocabulary — counting principles, $\binom{n}{k}$, the binomial theorem, Pascal's triangle, multinomial coefficients, the pigeonhole principle, and stars and bars. With them in hand, you're already equipped to read serious mathematics. The rest is practice.

Fundamental Counting Principles (Handout 1)

The Two Basic Counting Principles

Addition Principle: If a task can be done by A in m ways *or* by B in n ways (mutually exclusive), there are $m + n$ ways total.

Multiplication Principle: If a task consists of step A (m ways) *followed by* step B (n ways), there are $m \times n$ ways total.

Factorials, Permutations, Combinations

Factorial: $n! = n \times (n-1) \times \cdots \times 2 \times 1$, with $0! = 1$.

Permutations (order matters): the number of ways to arrange k items from n distinct items is

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

Combinations (order doesn't matter): the number of ways to choose k items from n distinct items is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}.$$

Pascal's Triangle and Interpretations (Handout 2)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The same number $\binom{n}{k}$ counts four different things:

1. the number of k-element subsets of an n-element set;
2. the number of grid paths from $(0,0)$ to $(k,\ n-k)$ using only right and up moves;
3. the number of binary strings of length n with exactly k ones;
4. the entry in row n, position k of Pascal's triangle.

The Binomial Theorem (Handout 3)

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n.$$

Special cases:

- $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$
- $(1-x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$

Properties of Pascal's Triangle (Handout 4)

Properties 1–4

- **Symmetry:** $\binom{n}{k} = \binom{n}{n-k}$
- **Row sum:** $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- **Alternating sum:** $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad (n \geq 1)$
- **Hockey stick:** $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$

Methodological themes: Handout 2 introduced *bijective proof* (show that two collections are in one-to-one correspondence). Handout 4 introduced *double counting* (count one collection two ways). Together with *complementary counting* (count the bad cases and subtract them from the total), which you have used since Handout 1, these are the three workhorse meta-methods of combinatorics — often more important than any individual formula.

Pascal's Triangle II and Advanced Counting (Handout 5)

Pigeonhole Principle

If n objects are placed into k boxes and $n > k$, then at least one box contains more than one object.
Generalized: if n objects are placed into k boxes, some box contains at least $\lceil n/k \rceil$ objects.

Stars and Bars

The number of ways to distribute n identical objects into k distinct bins:

- with empty bins allowed: $\binom{n+k-1}{k-1}$;
- with each bin getting at least one object: $\binom{n-1}{k-1}$.

More Properties of Pascal's Triangle

- **Even and odd subsets** (Property 3 again): toggling one element pairs every even-sized subset with an odd-sized one.
- **Powers of 11** (Property 5): row n gives the digits of 11^n; from row 5 on, carry.
- **Odd entries** (Property 6): $\binom{n}{k}$ is odd exactly when k fits under n in binary — the Sierpinski triangle.

Multinomial Coefficients

Splitting n people into named groups of sizes $k_1, \ldots, k_r$: $\binom{n}{k_1, \ldots, k_r} = \dfrac{n!}{k_1! \cdots k_r!}$ ways. The same number counts arrangements of letters with repeats (`MISSISSIPPI`) and is the coefficient of $x_1^{k_1} \cdots x_r^{k_r}$ in $(x_1 + \cdots + x_r)^n$.

Quick Check

1. How many different arrangements of the letters of `BANANA` are there?
2. In how many ways can 7 identical candies be given to 3 children (some may get none)?
3. What is the coefficient of x^3 in $(1+x)^8$?

Closing the combinatorics unit. Across these five handouts you have learned: the fundamental counting principles (addition and multiplication), factorials and permutations, binomial coefficients and their four interpretations, Pascal's triangle and its properties (symmetry, row sums, alternating sums, the hockey stick, powers of 11, the Sierpinski pattern), the binomial theorem, multinomial coefficients, and two powerful techniques — pigeonhole and stars and bars. You've also met two unifying methods that pervade combinatorics — *bijective proof* and *double counting* — which appear over and over in any further mathematics you'll meet. Counting will come back right away: the number of rows in a truth table is a counting question.

Introduction to Logic: What is Logic?

Logic is the study of correct reasoning. In mathematics, we use logic to determine whether statements are true or false and to construct valid proofs. We begin with **propositional logic**, the foundation of mathematical reasoning.

Today we start a new unit, on logic. It follows naturally from the combinatorics we just finished: in that unit we *proved* a lot of things, and now we step back and ask *what does it actually mean to prove something?* The arc of the unit:

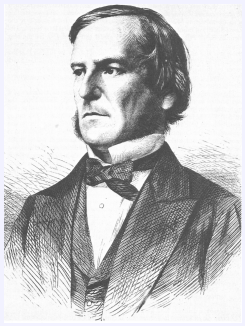
- **Today:** the basic operations of logic — NOT, AND, OR — together with truth tables and a first batch of algebraic identities.
- **Next:** more operations, including XOR ("exclusive or") and NAND, and the idea of expressing every logical function in terms of just a few primitives.
- **Then:** conditionals ("if … then …") and the biconditional — the operations that power mathematical statements like "$x > 0$ implies $x^2 > 0$."
- Our first careful introduction to mathematical *proof*, followed by the major methods of proof — direct proof, contrapositive, contradiction, cases, induction.
- *Quantifiers* — the symbols "for all" ($\forall$) and "there exists" ($\exists$) that let us talk about whole families of statements at once.
- A review of the whole logic unit.

The first handouts are about logical *operations* on statements; the later ones build on that foundation to do real mathematical reasoning. By the end of the unit, you should be able to read and write proofs at the level of an early college-level math course.

In logic, we work with **propositions**—statements that are either **true** (T) or **false** (F), but never both. These are also called **Boolean values**, named after mathematician George Boole.

Historical Note

The branch of mathematics this handout introduces — a logic in which every statement is precisely true or false, and operations combine them like arithmetic — is called **Boolean algebra** after the English mathematician **George Boole**. Boole was largely self-taught: born to a working-class family, he ran a school by age 19 and rose to a professorship at Queen's College Cork without a university degree. In two landmark works — *The Mathematical Analysis of Logic* (1847) and *An Investigation of the Laws of Thought* (1854) — he turned Aristotle's verbal syllogisms into algebraic equations, using $\times$ for AND and $+$ for inclusive OR.
For almost a century Boole's work sat as a curiosity in pure mathematics. Then in 1937, a 21-year-old MIT graduate student named Claude Shannon realised that Boolean algebra was *exactly* the right language for electrical switching circuits — and the digital age began. Boole's mathematics had been waiting eighty years for the hardware to catch up.



George Boole (1815–1864)

Example 1. Which of these are propositions?

1. "$2 + 2 = 4$" $\rightarrow$ Yes, this is a proposition (it's true).
2. "Paris is the capital of France" $\rightarrow$ Yes, this is a proposition (it's true).
3. "$5 > 10$" $\rightarrow$ Yes, this is a proposition (it's false).
4. "What time is it?" $\rightarrow$ No, questions are not propositions.
5. "$x + 3 = 7$" $\rightarrow$ No, the truth depends on the value of x.
6. "Close the door" $\rightarrow$ No, commands are not propositions.

We typically use capital letters like A, B, C, P, Q to represent propositions. For example:

- A: "It is raining"
- B: "The ground is wet"

Basic Logic Operations

Just as arithmetic has operations like addition and multiplication, logic has its own operations that combine propositions.

NOT (Negation)

The **NOT** operation (also written as $\neg A$ or $\text{NOT } A$) reverses the truth value of a proposition.

A	$\neg A$
T	F
F	T

Example 2. If A is "It is raining," then $\neg A$ is "It is **not** raining."
If A is true (it is raining), then $\neg A$ is false. If A is false (it is not raining), then $\neg A$ is true.

AND (Conjunction)

The **AND** operation (also written as $A \wedge B$ or $A \text{ AND } B$) is true only when **both** propositions are true.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3. Let A be "I finished my homework" and B be "I cleaned my room."

Then $A \wedge B$ means "I finished my homework **and** I cleaned my room."

This statement is true only if you did both things.

OR (Disjunction)

The **OR** operation (also written as $A \vee B$ or $A \text{ OR } B$) is true when **at least one** of the propositions is true.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

OR in Logic vs. OR in English

In everyday English, the word "or" is ambiguous—it can mean two different things:
Inclusive OR: "At least one is true" (possibly both)

- "Students who have taken calculus or physics may enroll."
- Here, students who took *both* courses can certainly enroll!

Exclusive OR: "Exactly one is true" (not both)

- "Would you like soup or salad with your meal?"
- The waiter typically means you choose one, not both.

In logic and mathematics, OR always means inclusive OR. The statement $A \vee B$ is true whenever at least one of A or B is true—including when both are true.
Why do mathematicians choose inclusive OR? Because it's more useful for stating theorems and conditions. When a theorem says "If $x > 0$ or $x < 0$, then $x^2 > 0$," we don't need to worry about whether both conditions might somehow hold—we just need at least one.
If you ever need exclusive OR (exactly one true), logic has a separate operation called **XOR**, which we'll study in the next handout.

Example 4. Consider: "To get a discount, you must be a student or a senior citizen."

Let S = "you are a student" and C = "you are a senior citizen."

In logic: $S \vee C$ is the condition for the discount.

What if someone is both a student AND a senior citizen? In logic, $S \vee C$ is still TRUE (both S and C are true, so "at least one" is satisfied). They get the discount!

This matches our intuition—being "extra qualified" shouldn't disqualify you.

Quick Check

4. If A is true and B is false, what is the truth value of $A \wedge B$?
5. If A is false and B is false, what is the truth value of $A \vee B$?
6. What is the truth value of $\neg(\neg A)$ if A is true?

Truth Tables

A **truth table** shows the truth value of a compound proposition for all possible combinations of truth values of its components. This is our main tool for analyzing logical expressions.

Example 5. Build a truth table for $\neg A \vee B$:

A	B	$\neg A$	$\neg A \vee B$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We first computed $\neg A$ for each row, then used those values with B to compute $\neg A \vee B$.

Example 6. Build a truth table for $(A \wedge B) \vee C$:
With three variables, we need $2^3 = 8$ rows:

A	B	C	$A \wedge B$	$(A \wedge B) \vee C$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Logic Laws

Two logical expressions are **logically equivalent** (written $\equiv$) if they have the same truth value for all possible inputs. We can verify equivalence using truth tables. (You may also see the symbol $\iff$ used for a closely related idea — the *biconditional connective* — but that's a topic for a later handout. For now, $\equiv$ is our equivalence symbol.)

Fundamental Laws

Many of the laws below have arithmetic analogues. The commutative laws look like $a + b = b + a$ and $ab = ba$; the associative laws look like $(a + b) + c = a + (b + c)$; the distributive law for AND over OR looks like $a(b + c) = ab + ac$. Boole's great insight was that logic and arithmetic have the same algebraic shape — so much so that we sometimes think of AND as "multiplication" and OR as "addition" on the two-element set $\{T, F\} = \{1, 0\}$. The one place logic and arithmetic part company is the distributive law for OR over AND ($A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$), which has no arithmetic counterpart — "addition" here distributes over "multiplication" too, a quirk specific to Boolean algebra.

Commutative Laws:

$$A \wedge B \equiv B \wedge A$$
$$A \vee B \equiv B \vee A$$

Associative Laws:

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$
$$(A \vee B) \vee C \equiv A \vee (B \vee C)$$

Distributive Laws:

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

Double Negation Law:

$$\neg(\neg A) \equiv A$$

Identity Laws:

$$A \wedge T \equiv A$$
$$A \vee F \equiv A$$

Domination Laws:

$$A \vee T \equiv T$$
$$A \wedge F \equiv F$$

De Morgan's Laws

The most consequential laws in logic are **De Morgan's Laws**. They describe how negation interacts with AND and OR — and you will use them every time you write the negation of a complex condition. Programmers use De Morgan to refactor `if`-statements; lawyers use De Morgan (without naming it) to read contracts; mathematicians use De Morgan to negate hypotheses in proofs.

Concretely, before stating the laws, think about how you would negate the sentence "Alice is hungry **and** tired." The negation isn't "Alice is not hungry and not tired" — that would be much stronger than the original. The actual negation is "Alice is not hungry, **or** not tired" — it's enough for *one* of the two conditions to fail. So the negation of an AND becomes an OR. Symmetrically, the negation of "It's raining **or** snowing" is "It's not raining **and** not snowing" (both have to fail). De Morgan's laws capture this pattern in formal symbols:

De Morgan's Laws

$$\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$
$$\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$$

In words:

- NOT (A AND B) $=$ (NOT A) OR (NOT B)
- NOT (A OR B) $=$ (NOT A) AND (NOT B)

Negation flips AND $\leftrightarrow$ OR, and pushes inside.

Example 7. Verify the first De Morgan's Law using a truth table:

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The columns for $\neg(A \wedge B)$ and $(\neg A) \vee (\neg B)$ are identical, confirming the equivalence.

Example 8. Using De Morgan's Laws in everyday language:

The statement "I don't have both a pen and paper" is equivalent to "I don't have a pen, or I don't have paper."

The statement "It's not the case that I'm tired or hungry" is equivalent to "I'm not tired, and I'm not hungry."

Aside (set-theory parallel). If you've already met sets, you've seen De Morgan's laws once before — in the form

$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \qquad \overline{A \cup B} = \overline{A} \cap \overline{B},$$

where the bar denotes "complement" (the elements *not* in the set). The logical version above is the same law, with propositions in place of sets, "and" in place of intersection, "or" in place of union, and negation in place of complement. This is no coincidence: a proposition is essentially the indicator of a set — the set of worlds in which it is true. We covered the set version in Math 5 (handouts 25–26) and Math 6 (handouts 6–7); if you want a second perspective on what's happening here, those are good places to look.

Quick Check

7. Use De Morgan's Law to simplify $\neg(P \vee Q)$.
8. What is the negation of "Alice is smart and Bob is tall"?

Key Takeaways

- A **proposition** is a statement that is either true or false.
- **NOT** ($\neg$) reverses truth values: $\neg T = F$, $\neg F = T$.
- **AND** ($\wedge$) is true only when both parts are true.
- **OR** ($\vee$) is true when at least one part is true.
- **Truth tables** show all possible truth values of a compound proposition.
- **De Morgan's Laws**: $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$.

Common Mistakes

- **Confusing AND and OR**: AND requires *both* to be true; OR requires *at least one*.
- **Misapplying De Morgan**: When you negate, you must change AND to OR (and vice versa), AND negate each part.
- **Forgetting inclusive OR**: In logic, "A or B" includes the case where both are true.

Practice Problems

1. How many 4-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if:
   (a) Repetition is allowed?
   (b) Repetition is not allowed?
2. A committee of 5 people is to be formed from 8 men and 6 women. How many committees can be formed if:
   (a) There are no restrictions?
   (b) The committee must have exactly 3 men and 2 women?
   (c) The committee must have at least one woman?
3. How many ways are there to distribute 10 identical candies among 4 children if:
   (a) Each child can receive any number of candies (including zero)?
   (b) Each child must receive at least one candy?
4. Which of these are propositions? For each proposition, say whether it is true, false, or unknown.
   (a) "17 is a prime number."
   (b) "Is 17 a prime number?"
   (c) "$x^2 = 4$."
   (d) "Every even number greater than 2 is the sum of two primes."
   (e) "Please be quiet."
   (f) "This sentence has five words."
5. Let P be "It is sunny" and Q be "I go to the beach." Write each in plain English:
   (a) $P \wedge \neg Q$
   (b) $\neg P \vee Q$
   (c) $\neg(P \vee Q)$ (use De Morgan's law to make it sound natural)
6. Let A be "Ann is home" and B be "Ben is home." Write in symbols:
   (a) Ann and Ben are both home.
   (b) Neither Ann nor Ben is home.
   (c) At least one of them is not home.
   (d) Ann is home, but Ben isn't.
7. Suppose A is true, B is false, and C is true. Find the truth value of:
   (a) $A \wedge (B \vee C)$
   (b) $\neg A \vee (B \wedge C)$
   (c) $\neg(A \wedge B) \wedge \neg C$
8. How many rows does a truth table need for a formula with 4 variables? With 5? Explain using the multiplication principle.
9. Write truth tables for each expression:
   (a) $A \wedge \neg B$
   (b) $\neg A \vee \neg B$

(c) $(A \vee B) \wedge C$

10. For which truth values of A and B is $(A \vee B) \wedge \neg A$ true?
11. Use a truth table to verify that $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.
12. Simplify using the logic laws:
    (a) $A \vee (A \wedge B)$
    (b) $(A \wedge B) \vee (A \wedge \neg B)$
    (c) $\neg(\neg A \vee \neg B)$
13. Simplify using De Morgan's Laws:
    (a) $\neg(P \wedge Q)$
    (b) $\neg(\neg A \vee B)$
    (c) $\neg(A \wedge (B \vee C))$
14. Is $\neg(A \wedge B)$ equivalent to $\neg A \wedge \neg B$? If not, find truth values of A and B for which they differ.
15. Write the negation of each condition on numbers without using the word "not":
    (a) $x > 2$ and $x < 5$
    (b) $x \leq 0$ or $x \geq 1$
    (c) n is even and $n > 10$
16. Express the negation of each statement in simple English (without using "it is not the case that"):
    (a) "I will go to the store and buy milk."
    (b) "The number is positive or zero."
    (c) "It is raining and cold."
17. Write a formula in A, B, C that is true exactly when all three are false. Give two different answers.
18. Simplify the sentence "It is not true that I like neither apples nor bananas."
19. On the island of knights (who always tell the truth) and knaves (who always lie), A says: "B is a knight and I am a knave." What are A and B?

Practice Problems: Solutions

1. **4-digit numbers from digits 1–6:**
   (a) With repetition: $6^4 = 1296$
   (b) Without repetition: $P(6,4) = 6 \times 5 \times 4 \times 3 = 360$

2. **Committee of 5 from 8 men and 6 women:**
   (a) No restrictions: $\binom{14}{5} = 2002$
   (b) Exactly 3 men and 2 women: $\binom{8}{3} \times \binom{6}{2} = 56 \times 15 = 840$
   (c) At least one woman: Total $-$ All men $= 2002 - \binom{8}{5} = 2002 - 56 = 1946$

3. **10 candies among 4 children:**
   (a) Empty bins allowed: $\binom{10+4-1}{4-1} = \binom{13}{3} = 286$
   (b) At least one each: $\binom{10-1}{4-1} = \binom{9}{3} = 84$

4. (a) Proposition, true. (b) Not a proposition (a question). (c) Not a proposition — its truth depends on x. (d) A proposition, but nobody knows whether it is true! This is *Goldbach's conjecture* (1742), still unsolved. (e) Not a proposition (a command). (f) Proposition, true — count the words.

5. (a) "It is sunny and I don't go to the beach." (b) "It isn't sunny, or I go to the beach." (c) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$: "It isn't sunny, and I don't go to the beach."

6. (a) $A \wedge B$ (b) $\neg A \wedge \neg B$ (c) $\neg A \vee \neg B$ (d) $A \wedge \neg B$ ("but" means "and" in logic).

7. (a) $T \wedge (F \vee T) = T \wedge T = T$. (b) $F \vee (F \wedge T) = F \vee F = F$. (c) $\neg F \wedge \neg T = T \wedge F = F$.

8. Each variable can be T or F, independently, so by the multiplication principle there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ rows for 4 variables and $2^5 = 32$ rows for 5. Each new variable doubles the table.

9. Truth tables:

   (a) $A \wedge \neg B$:

A	B	$\neg B$	$A \wedge \neg B$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

   (b) $\neg A \vee \neg B$:

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

   (c) $(A \vee B) \wedge C$:

A	B	C	$A \vee B$	$(A \vee B) \wedge C$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

10. $\neg A$ must be true, so A is false; then $A \vee B$ forces B to be true. Only $A = F$, $B = T$.

11. Both $A \vee (B \wedge C)$ and $(A \vee B) \wedge (A \vee C)$ give the same truth table (T for rows where A is T, or where both B and C are T).

12. (a) A (if A is true, the whole thing is true; if A is false, so is $A \wedge B$). (b) By the distributive law, $A \wedge (B \vee \neg B) = A \wedge T = A$. (c) By De Morgan, $\neg\neg A \wedge \neg\neg B = A \wedge B$.

13. De Morgan simplifications:

    (a) $\neg(P \wedge Q) = \neg P \vee \neg Q$

    (b) $\neg(\neg A \vee B) = \neg(\neg A) \wedge \neg B = A \wedge \neg B$

    (c) $\neg(A \wedge (B \vee C)) = \neg A \vee \neg(B \vee C) = \neg A \vee (\neg B \wedge \neg C)$

14. No. Take $A = T$, $B = F$: then $\neg(A \wedge B) = \neg F = T$, but $\neg A \wedge \neg B = F \wedge T = F$. (By De Morgan, the correct partner is $\neg A \vee \neg B$.)

15. (a) $x \leq 2$ or $x \geq 5$. (b) $x > 0$ and $x < 1$, that is, $0 < x < 1$. (c) n is odd or $n \leq 10$.

16. Negations:

    (a) "I will not go to the store, or I will not buy milk."

    (b) "The number is negative." (i.e., not positive and not zero)

    (c) "It is not raining, or it is not cold."

17. $\neg A \wedge \neg B \wedge \neg C$, or, by De Morgan, $\neg(A \vee B \vee C)$.

18. With A = "I like apples" and B = "I like bananas": $\neg(\neg A \wedge \neg B) \equiv A \vee B$ — "I like apples or bananas (or both)."

19. If A were a knight, A's statement would be true — but it says A is a knave. So A is a **knave**, and the statement is false. Since "I am a knave" is true, the false part must be "B is a knight": B is a **knave** too.

Homework

1. How many different arrangements of the letters of `COMMITTEE` are there?
2. How many ways can you place 12 identical balls into 5 distinct boxes such that each box has at least 2 balls?
3. Which of these are propositions? For those that are, say whether they are true or false.
   (a) "100 is a perfect square."
   (b) "How old are you?"
   (c) "$2^{10} > 1000$."
   (d) "n is odd."
   (e) "There are more than a million prime numbers."
   (f) "Stop talking!"
4. Let R be "it is raining," W be "it is windy," and C be "it is cold." Write in symbols:
   (a) It is raining and windy, but not cold.
   (b) It is neither raining nor windy.
   (c) It is cold, or it is raining and windy.
   (d) It is not both raining and cold.
5. Write truth tables for:
   (a) $\neg(A \wedge B)$
   (b) $(A \vee B) \wedge \neg A$
   (c) $\neg A \wedge (B \vee C)$
6. Simplify using logic laws:
   (a) $\neg(\neg P \wedge \neg Q)$
   (b) $A \wedge (A \vee B)$
   (c) $(A \vee B) \wedge (\neg A \vee B)$
7. Find all truth values of A, B, C that make $(A \vee B) \wedge (\neg A \vee C) \wedge \neg C$ true.
8. Write the negation of each statement in simple English:
   (a) "The test is hard and the time is short."
   (b) "I will study math or physics tonight."
   (c) "She is smart and hardworking."
9. Simplify $\neg(A \vee \neg B) \vee (A \wedge B)$, naming the law you use at each step.
10. **Restaurant menu.** A restaurant menu says: *"The fixed-price dinner includes an entree, a dessert, and soup or salad."* Using the basic propositions
    - E: your dinner includes an entree
    - D: your dinner includes a dessert
    - P: your dinner includes a soup
    - S: your dinner includes a salad

    write this menu policy as a logical formula. (Be careful: the "or" here is inclusive — some restaurants let you get both, others won't — but in standard English menu phrasing it's read as "at least one of soup or salad.")

11. A door lock opens when you enter the correct code *and* either swipe your card or scan your fingerprint. Let K be "correct code entered," D be "card swiped," and F be "fingerprint scanned."

    (a) Write a formula for "the lock opens."

    (b) Write a formula for "the lock stays shut," and use De Morgan's laws to push the negation all the way inside. Say the result in plain English.

12. H **Knights and Knaves:** On the island of knights and knaves, knights always tell the truth and knaves always lie. You meet two people, Bart and Ted. Bart claims: "I and Ted are both knights or both knaves." Ted tells you: "Bart would tell you that I am a knave." Who is a knight and who is a knave?

13. M **Knights and Knaves.** Three islanders A, B, C. A says: "B and C are both knights." B says: "A is a knave or C is a knave." C says: "B is a knave." Who is a knight and who is a knave?

14. M **Majority vote.** Three judges A, B, C vote yes (T) or no (F), and a proposal passes when at least two of them vote yes.

    (a) Write a formula using $\wedge$, $\vee$ that is true exactly when the proposal passes.

    (b) Show that $(A \vee B) \wedge (A \vee C) \wedge (B \vee C)$ is another answer.

Quick Check Answers

1. $\dfrac{6!}{3!\,2!\,1!} = 60$ (three A's, two N's, one B).
2. Stars and bars: $\binom{7+3-1}{3-1} = \binom{9}{2} = 36$.
3. $\binom{8}{3} = 56$.
4. $A \wedge B = T \wedge F = F$
5. $A \vee B = F \vee F = F$
6. $\neg(\neg A) = \neg(\neg T) = \neg F = T$
7. $\neg(P \vee Q) = \neg P \wedge \neg Q$
8. "Alice is not smart, or Bob is not tall."