

MATH 8

PASCAL'S TRIANGLE II AND ADVANCED COUNTING — SHORT 2026–27

We finish our tour of Pascal's triangle with a counting proof, two surprises, and a new kind of coefficient. Then we add two powerful counting tools: the pigeonhole principle and stars and bars.

Even and Odd Subsets

In Handout 4 we proved that the alternating row sums are 0: for $n \geq 1$, $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots = 0$. In words, *a set with at least one element has as many even-sized subsets as odd-sized ones*. Here is why, with no algebra at all. Pick one element x and **toggle** it: put x into a subset that doesn't have it, take it out of one that does. Toggling changes the size by exactly one, so it pairs every even-sized subset with an odd-sized one — and toggling twice brings you back. So the two kinds come in matching pairs.

Powers of 11

Read the short rows of Pascal's triangle as numbers:

$$1, \quad 11, \quad 121, \quad 1331, \quad 14641 \quad \text{— these are } 11^0, 11^1, 11^2, 11^3, 11^4$$

The reason: $11^n = (10 + 1)^n$, and the binomial theorem puts the entries of row n in front of the powers of 10. From row 5 on, some entries have two digits, so we carry:

$$11^5 = 1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10 + 1 = 161051.$$

Odd Entries and the Sierpinski Triangle

Color the odd entries of Pascal's triangle (try it on the Pascal's triangle page at the end): triangles inside triangles appear — the famous **Sierpinski triangle**. There is a secret rule behind it, found by Édouard Lucas in 1878:

The binary-digit rule

Write n and k in binary. $\binom{n}{k}$ is **odd** exactly when k fits under n : wherever k has a 1, n has a 1 too.

Example. $13 = 1101_2$. The numbers that fit under it are 0, 1, 4, 5, 8, 9, 12, 13 — and these are exactly the odd entries of row 13. The rows that are *entirely* odd are those whose binary digits are all 1s: rows 1, 3, 7, 15, 31, ...

Splitting a Group into Teams: Multinomials

Example. Split 6 students into teams Red, Blue, Green of 2 each. Choose Red ($\binom{6}{2} = 15$ ways), then Blue from the remaining 4 ($\binom{4}{2} = 6$), and the last 2 are Green: $15 \cdot 6 \cdot 1 = 90$. As factorials, everything in the middle cancels:

$$\binom{6}{2} \binom{4}{2} \binom{2}{2} = \frac{6!}{2!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!0!} = \frac{6!}{2!2!2!} = 90.$$

In general, splitting n people into named groups of sizes $k_1, \dots, k_r$ can be done in $\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_1! \cdots k_r!}$ ways — the **multinomial coefficient**. (If the groups have no names, divide by the number of ways to hand out the names.)

Why MISSISSIPPI is the same problem. Write each student's team letter in a row: a split becomes a word like RBGRGB. So arranging the letters RRBBGG is splitting into teams. Likewise, arranging MISSISSIPPI splits its 11 positions into teams M (1), I (4), S (4), P (2): $\frac{11!}{1! 4! 4! 2!} = 34,650$. And in $(a + b + c)^n$, the coefficient of $a^{k_1} b^{k_2} c^{k_3}$ is $\binom{n}{k_1, k_2, k_3}$: each term splits the n factors into an a -team, a b -team, and a c -team.

The Pigeonhole Principle

Pigeonhole principle

If more than k objects are put into k boxes, some box gets at least two. More generally, if n objects go into k boxes, some box gets at least $\lceil n/k \rceil$.

Example. Among any 13 people, two were born in the same month: 13 people, 12 months. The principle proves something *must* happen — without telling us which two people it is. The art is choosing the right “boxes.”

Stars and Bars

How many ways can you give 10 identical candies to 3 children? Draw the candies as stars and separate the children's shares with 2 bars:

$$*** | ***** | ** \quad (3, 5, 2).$$

Every distribution is a row of 10 stars and 2 bars, and every such row is a distribution. So we just choose where the 2 bars go among 12 places: $\binom{12}{2} = 66$.

Stars and bars

n identical objects into k distinct boxes: $\binom{n+k-1}{k-1}$ ways (empty boxes allowed).

Common Mistakes

- **Forgetting to carry** with 11^n from row 5 on.
- **Named vs. unnamed groups:** divide by the ways to hand out names when the groups are interchangeable.
- **Stars and bars needs identical objects:** distinct objects are counted differently.

Homework

1. Compute 101^5 . Then compute 1001^3 .
2. **Multinomial practice.** Compute:
 - (a) The coefficient of $x^2y^3z^2$ in $(x + y + z)^7$.
 - (b) The number of distinct anagrams of the word COMBINATORICS.
 - (c) The coefficient of a^2b^2c in $(2a + 3b - c)^5$.
3. Nine different books are to be given to three students, three books each.
 - (a) In how many ways can this be done?
 - (b) Find the coefficient of $x^3y^3z^3$ in $(x + y + z)^9$. Why is it the same number?
4. How many 4-digit numbers have digits that add up to 9? (For example, 1206 and 9000. Remember that the first digit can't be 0.)
5. How many ways can you distribute 20 identical candies among 4 children if each child must get at least 2 candies?
6. If 6 points are placed inside an equilateral triangle of side 2, prove that at least two points are within distance 1 of each other.
7. Six different numbers are chosen from $1, 2, \dots, 10$. Prove that two of them add up to 11. Is the same true for five numbers?
8. **H A sum divisible by 10.** Prove that among any 10 integers $a_1, a_2, \dots, a_{10}$, some of them (one or more, standing next to each other in the list) have a sum divisible by 10. (*Hint: look at the ten sums a_1 , $a_1 + a_2$, $a_1 + a_2 + a_3$, $\dots$, and their remainders when divided by 10.*)
9. **Class presentations.** A class of 20 students must each pick one weekday (Monday through Friday) to give a 10-minute presentation. The teacher only cares *how many* presentations land on each day, not who gives them. How many distinct schedules are possible?
10. **Four groups of unequal size.** A teacher wants to split a class of 19 students into four working groups of sizes 5, 5, 5, 4. The groups are not labeled — only the membership matters — but the three size-5 groups are themselves interchangeable. How many ways can this be done?
11. **Rolling n dice.** You roll n ordinary six-sided dice simultaneously.
 - (a) If the dice are distinguishable (it matters which die shows which face), how many outcomes are possible?
 - (b) If the dice are indistinguishable (you only see the multiset of faces), how many outcomes are possible?

Pascal's Triangle (rows 0–20; row number on the left)

0																					
1																					
2																					
3																					
4																					
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1																				
1	1																			
1	2	1																		
1	3	3	1																	
1	4	6	4	1																
1	5	10	10	5	1															
1	6	15	20	15	6	1														
1	7	21	35	35	21	7	1													
1	8	28	56	70	56	28	8	1												
1	9	36	84	126	126	84	36	9	1											
1	10	45	120	210	252	210	120	45	10	1										
1	11	55	165	330	462	462	330	165	55	11	1									
1	12	66	220	495	792	924	792	495	220	66	12	1								
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1							
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1						
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1					
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1				
1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	136	17	1			
1	18	153	816	3060	8568	18564	31824	43758	48620	43758	31824	18564	8568	3060	816	153	18	1		
1	19	171	969	3876	11628	27132	50388	75582	92378	92378	75582	50388	27132	11628	3876	969	171	19	1	
1	20	190	1140	4845	15504	38760	77520	125970	167960	184756	167960	125970	77520	38760	15504	4845	1140	190	20	1

Pascal's Triangle (rows 0–20; row number on the left)

0																						1																					
1																					1		1																				
2																				1		2		1																			
3																			1		3		3		1																		
4																		1		4		6		4		1																	
5																	1		5		10		10		5		1																
6																1		6		15		20		15		6		1															
7															1		7		21		35		35		21		7		1														
8														1		8		28		56		70		56		28		8		1													
9													1		9		36		84		126		126		84		36		9		1												
10												1		10		45		120		210		252		210		120		45		10		1											
11											1		11		55		165		330		462		462		330		165		55		11		1										
12										1		12		66		220		495		792		924		792		495		220		66		12		1									
13									1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1								
14								1		14		91		364		1001		2002		3003		3432		3003		2002		1001		364		91		14		1							
15							1		15		105		455		1365		3003		5005		6435		6435		5005		3003		1365		455		105		15		1						
16						1		16		120		560		1820		4368		8008		11440		12870		11440		8008		4368		1820		560		120		16		1					
17					1		17		136		680		2380		6188		12376		19448		24310		24310		19448		12376		6188		2380		680		136		17		1				
18				1		18		153		816		3060		8568		18564		31824		43758		48620		43758		31824		18564		8568		3060		816		153		18		1			
19			1		19		171		969		3876		11628		27132		50388		75582		92378		92378		75582		50388		27132		11628		3876		969		171		19		1		
20		1		20		190		1140		4845		15504		38760		77520		125970		167960		184756		167960		125970		77520		38760		15504		4845		1140		190		20		1	