

## MATH 8

### PASCAL'S TRIANGLE II AND ADVANCED COUNTING — FULL 2026–27

## Introduction

This handout has two halves. In the first, we finish our tour of Pascal's triangle. In Handout 4 we proved four of its properties — symmetry, row sums, alternating row sums, and the hockey stick — mostly by *double counting*: counting one collection in two different ways. We start by revisiting Property 3 with a counting proof. Then come two more properties, each surprising in its own way: a link with the powers of 11 (Property 5), and a fractal hiding among the odd entries (Property 6). Finally, we go beyond binomials to the *multinomial* coefficients.

In the second half we add two new counting techniques to our toolkit:

- **The pigeonhole principle** — a deceptively simple existence tool used to prove that some configuration *must* occur.
- **Stars and bars** — counting distributions of *identical* objects into distinct bins.

### Historical Note

Many properties of Pascal's triangle are visible at a glance — symmetry, row sums, alternating sums — but some of its patterns are hidden much deeper. The French mathematician **Édouard Lucas** found one of the most surprising. In an 1878 paper he proved what is now called *Lucas's theorem*: the remainder of  $\binom{n}{k}$  when divided by a prime  $p$  can be read off from the digits of  $n$  and  $k$  written in base  $p$ . You will see what it says for  $p = 2$  — odd and even entries — in Property 6, where it explains a famous fractal hiding in Pascal's triangle.

Lucas had a wide-ranging mathematical life: he invented the *Tower of Hanoi* puzzle, studied the sequence now known as the *Lucas numbers* (cousins of the Fibonacci numbers), and proved that the Mersenne number  $2^{127} - 1$  is prime — the largest prime found by hand in the pre-computer era.



Édouard Lucas  
(1842–1891)

## Property 3 Revisited: Even and Odd Subsets

In Handout 4 we proved that the alternating row sums are 0 (for  $n \geq 1$ ) by algebra and by watching how the triangle is built. In Handout 4's homework you discovered a third, counting proof. Here it is written out.

### Alternating Sum of Row $n$ (for $n \geq 1$ )

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

### Proof: Combinatorial

The alternating sum counts (subsets of even size) – (subsets of odd size).

**Claim:** For any non-empty set, the number of even-sized subsets equals the number of odd-sized subsets.

**Proof of claim:** Fix one element  $x$  in our  $n$ -element set. Create a pairing: match each subset  $S$  with the subset obtained by *toggling*  $x$ : add  $x$  if it is not in  $S$ , remove it if it is. This switches the parity of the subset size, giving a perfect matching between even-sized and odd-sized subsets.  $\square$

**Example 1.** For the set  $\{a, b, c\}$ , the subsets are:

- Even-sized:  $\emptyset, \{a, b\}, \{a, c\}, \{b, c\}$  — 4 subsets
- Odd-sized:  $\{a\}, \{b\}, \{c\}, \{a, b, c\}$  — 4 subsets

Indeed,  $\binom{3}{0} + \binom{3}{2} = 1 + 3 = 4$  and  $\binom{3}{1} + \binom{3}{3} = 3 + 1 = 4$ .

## Property 5: Powers of 11

You met this one in the homework of Handout 4. It is more of a curiosity than a tool, but a charming one. Read each short row of Pascal's triangle as a multi-digit number (1, 11, 121, 1331, 14641, ...) and you get the powers of 11:

| Row | Entries       | Number         |
|-----|---------------|----------------|
| 0   | 1             | $1 = 11^0$     |
| 1   | 1, 1          | $11 = 11^1$    |
| 2   | 1, 2, 1       | $121 = 11^2$   |
| 3   | 1, 3, 3, 1    | $1331 = 11^3$  |
| 4   | 1, 4, 6, 4, 1 | $14641 = 11^4$ |

This works because  $(10 + 1)^n = 11^n$ , and the binomial expansion gives the entries of Pascal's Triangle as coefficients of powers of 10!

Starting with row 5, some coefficients have more than one digit, so simply concatenating them no longer works. Instead, treat the coefficients as place values and carry in base 10. Row 5 gives

$$1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10 + 1 = 161051 = 11^5.$$

Thus the binomial expansion still produces  $11^n$ , but carrying is now part of reading the answer.

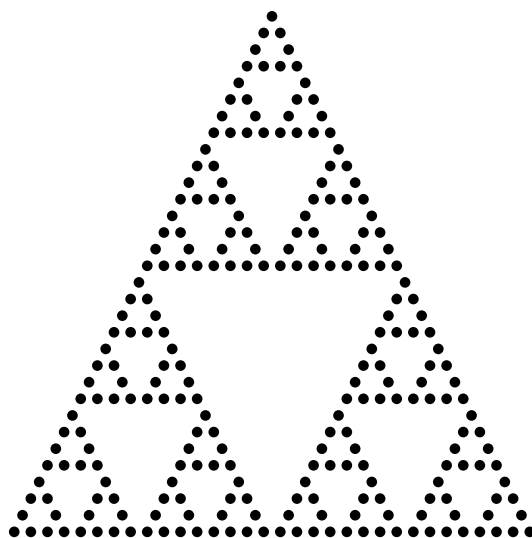
### Quick Check

1. Why can't you read  $11^5$  directly off row 5 (1, 5, 10, 10, 5, 1)? What is  $11^5$ ?

## Property 6: All-Odd Rows and the Sierpinski Pattern

For the final property we go visual — and explain the picture you colored in the homework of Handout 4. So far we've cared about the values of binomial coefficients; now we ignore the values and just ask one question of each entry: is it odd or even? The answer produces one of the most famous fractal pictures in all of mathematics.

Mark each entry of Pascal's Triangle with a  $\bullet$  if it is *odd*, and leave it blank if it is *even*. The resulting picture is the famous **Sierpinski triangle** fractal — one of the most striking visual surprises in elementary mathematics.



## A Secret Rule in Binary

Can we tell whether an entry is odd *without* computing it? Surprisingly, yes — and the trick is to write the numbers in binary. This is the case  $p = 2$  of Lucas's theorem (see the historical note at the start of the handout).

### The binary-digit rule

Write  $n$  and  $k$  in binary, one above the other, lined up on the right. Then  $\binom{n}{k}$  is **odd** exactly when  $k$  fits under  $n$ : wherever  $k$  has a 1,  $n$  also has a 1 in the same place.

**Example 2.** Take row 13. In binary,  $13 = 1101_2$ .

$$\begin{array}{rcccc} n = 13: & 1 & 1 & 0 & 1 \\ k = 5: & 0 & 1 & 0 & 1 \end{array} \qquad \begin{array}{rcccc} n = 13: & 1 & 1 & \textcolor{red}{0} & 1 \\ k = 6: & 0 & 1 & \textcolor{red}{1} & 0 \end{array}$$

On the left, every 1 of  $k = 5$  sits under a 1 of 13, so  $\binom{13}{5}$  should be odd — and indeed  $\binom{13}{5} = 1287$ . On the right,  $k = 6$  has a 1 (in red) where 13 has a 0, so  $\binom{13}{6}$  should be even — and indeed  $\binom{13}{6} = 1716$ .

The numbers that fit under  $1101_2$  are

$$0000, 0001, 0100, 0101, 1000, 1001, 1100, 1101, \quad \text{that is, } k = 0, 1, 4, 5, 8, 9, 12, 13.$$

These are exactly the odd entries of row 13. Find row 13 in the picture and check!

We won't prove the rule here — Lucas's proof needs more number theory than we have yet — but you can test it on any row of the picture.

## Which Rows Are Entirely Odd?

Look at the picture again: some rows are completely filled. They are rows 0, 1, 3, 7, 15, 31, ... — one less than a power of 2.

### Theorem

Row  $n$  of Pascal's triangle is all-odd if and only if  $n = 2^j - 1$  for some  $j \geq 0$ .

The binary-digit rule makes this easy to see. The numbers 1, 3, 7, 15, 31, ... are exactly the numbers whose binary digits are *all* 1s:  $1_2, 11_2, 111_2, 1111_2, 11111_2, \dots$  (and row 0 is just the single entry 1).

- If every binary digit of  $n$  is 1, then every  $k$  from 0 to  $n$  fits under  $n$  — there is a 1 above every digit of  $k$ . So the whole row is odd.
- If  $n$  has a 0 somewhere, take  $k$  to be the number with a single 1 in that place. Then  $k$  does not fit under  $n$ , so  $\binom{n}{k}$  is even. For example,  $12 = 1100_2$  has a 0 in the 1s place, so  $\binom{12}{1} = 12$  is even.

## Where the Fractal Comes From

The same rule explains the triangles within triangles. Look at rows 8 to 15: in binary each of them is a 1 in the 8s place followed by the digits of a row from 0 to 7 (for example,  $13 = 1101_2$  is “8+ row 5”). So in rows 8 to 15:

- the entries with  $k$  from 0 to 7 repeat the pattern of rows 0 to 7 — the *left* copy;
- the entries with  $k$  from 8 on also repeat that pattern, because  $k$  uses the 8s place too — the *right* copy;
- everything in between is even — the empty triangle in the middle.

The same thing happens with rows 16 to 31 (three copies of rows 0 to 15), and so on forever. That is why the picture looks the same at every scale.

### Quick Check

2. Which of rows 15, 16, and 31 of Pascal’s triangle are entirely odd?
3. Is  $\binom{6}{3}$  odd or even? Decide using the binary-digit rule, then check by computing it.

## Beyond Binomials: Multinomial Coefficients

The binomial coefficient  $\binom{n}{k}$  counts the ways to split a group of  $n$  people into *two* parts: the  $k$  who are chosen and the  $n - k$  who are not. What if we split a group into *three* or more parts?

**Example 3** (Three teams of 2). Six students must be split into three teams of two: Red, Blue, and Green. In how many ways?

**Solution:** Fill the teams one at a time.

- Choose the 2 Red players from all 6 students:  $\binom{6}{2} = 15$  ways.
- Choose the 2 Blue players from the 4 students who are left:  $\binom{4}{2} = 6$  ways.
- The last 2 students form the Green team:  $\binom{2}{2} = 1$  way.

By the multiplication principle there are  $15 \cdot 6 \cdot 1 = 90$  ways. Look what happens when we write the binomial coefficients as factorials:

$$\binom{6}{2} \binom{4}{2} \binom{2}{2} = \frac{6!}{2!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!0!} = \frac{6!}{2!2!2!} = \frac{720}{8} = 90.$$

The red 4!s cancel, and so do the blue 2!s (and  $0! = 1$ ). What’s left is  $6!$  divided by the factorials of the three team sizes.

**Example 4** (Teams of 1, 2, and 3). Now split the six students into a team of 1 (the reporter), a team of 2 (the artists), and a team of 3 (the builders).

**Solution:** Choose the reporter, then the artists from the remaining 5, then the builders are the last 3:

$$\binom{6}{1} \binom{5}{2} \binom{3}{3} = 6 \cdot 10 \cdot 1 = 60 = \frac{6!}{1!2!3!}.$$

We could fill the teams in a different order — builders first, then artists, then the reporter:  $\binom{6}{3}\binom{3}{2}\binom{1}{1} = 20 \cdot 3 \cdot 1 = 60$ . Same answer, as it must be.

The same cancellation happens every time, so:

### Multinomial coefficient

The number of ways to split  $n$  people into named groups of sizes  $k_1, k_2, \dots, k_r$  (with  $k_1 + k_2 + \dots + k_r = n$ ) is

$$\binom{n}{k_1, k_2, \dots, k_r} = \frac{n!}{k_1! k_2! \dots k_r!}.$$

With two groups this is the familiar binomial coefficient:  $\binom{n}{k, n-k} = \frac{n!}{k! (n-k)!} = \binom{n}{k}$ .

**Named or not?** In the first example the teams had names (Red, Blue, Green). If the three pairs had no names, then every split would have been counted  $3! = 6$  times — once for each way of handing out the three names — so there would be only  $90/6 = 15$  ways.

### Why MISSISSIPPI Is the Same Problem

Line up the six students in alphabetical order and write down, for each one, the first letter of their team. A split into Red, Blue, and Green teams becomes a six-letter word with two R's, two B's, and two G's. For example,

R B G R G B

means: student 1 is on Red, student 2 on Blue, student 3 on Green, student 4 on Red, and so on. Every split gives exactly one word, and every word gives exactly one split. So the number of splits equals the number of ways to arrange the letters RRBGGG:

$$\frac{6!}{2! 2! 2!} = 90.$$

Now read it backwards. An arrangement of MISSISSIPPI is a split of the 11 *positions* into four teams: 1 position gets M, 4 get I, 4 get S, and 2 get P. Filling the teams one at a time, exactly as with the students:

$$\underbrace{\binom{11}{1}}_M \underbrace{\binom{10}{4}}_{I's} \underbrace{\binom{6}{4}}_{S's} \underbrace{\binom{2}{2}}_{P's} = 11 \cdot 210 \cdot 15 \cdot 1 = 34,650 = \frac{11!}{1! 4! 4! 2!}.$$

Splitting people into teams and arranging letters with repeats are the same problem: the letters are the team names, and the positions are the people.

### The Multinomial Theorem

Expanding a power of a sum of three or more terms is a team-splitting problem too. In

$$(a + b + c)^n = \underbrace{(a + b + c)(a + b + c) \dots (a + b + c)}_{n \text{ factors}},$$

each term of the expansion picks one letter from each factor — that is, it splits the  $n$  factors into an “ $a$  team,” a “ $b$  team,” and a “ $c$  team.” So the coefficient of  $a^{k_1} b^{k_2} c^{k_3}$  is the number of such splits,  $\binom{n}{k_1, k_2, k_3}$ .

### The Multinomial Theorem

The coefficient of  $x_1^{k_1} x_2^{k_2} \dots x_r^{k_r}$  in  $(x_1 + x_2 + \dots + x_r)^n$  is  $\binom{n}{k_1, k_2, \dots, k_r}$ , for any  $k_1 + k_2 + \dots + k_r = n$ .

**Example 5** (Trinomial expansion). Expand  $(a + b + c)^3$ .

**Solution:** The terms come from all ways to split the 3 factors into teams:

$$\begin{aligned}(a + b + c)^3 &= a^3 + b^3 + c^3 \\ &\quad + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 \\ &\quad + 6abc.\end{aligned}$$

For example, the coefficient of  $a^2b$  is  $\frac{3!}{2!1!0!} = 3$ , and the coefficient of  $abc$  is  $\frac{3!}{1!1!1!} = 6$ : one factor for each letter, in any order.

### Quick Check

4. How many distinct anagrams does the word BANANA have?
5. Find the coefficient of  $a^2bc$  in  $(a + b + c)^4$ .

## Two Counting Techniques

Everything so far has come from two basic moves — the multiplication principle and the binomial coefficients  $\binom{n}{k}$  — together with Pascal’s triangle and the binomial theorem. These take us a long way, but they leave many natural counting problems out of reach. The next two techniques fill the gap. You will use them again and again — in number theory, in probability, and in competition problems. We start with pigeonhole, the easiest to state and the one that appears most often later.

### Technique 1: The Pigeonhole Principle

#### Historical Note

The Pigeonhole Principle is so obvious that it feels like it shouldn’t need a name — yet it is one of the most surprisingly powerful tools in all of mathematics. It is named after the German mathematician **Peter Gustav Lejeune Dirichlet**, who called it the *Schubfachprinzip* (“drawer principle”) — imagine sorting socks into drawers. Dirichlet wasn’t the first to notice it: back in 1622 the French writer Jean Leurechon observed that two people must have exactly the same number of hairs on their heads! But Dirichlet turned it into a serious tool. In 1842 he used it to prove that every irrational number  $\alpha$  has infinitely many fractions  $\frac{p}{q}$  with  $\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$  — approximations that are amazingly accurate for the size of their denominators (*Dirichlet’s approximation theorem*).

Dirichlet was a giant of 19th-century mathematics. He proved that there are infinitely many primes in any arithmetic progression  $a, a+d, a+2d, \dots$  (provided  $a$  and  $d$  have no common factor), introduced the modern definition of a function, and inspired a whole generation of German mathematicians including Riemann and Dedekind. His pigeonhole argument has since become a staple of combinatorics, computer science, and competition mathematics.



P. G. L. Dirichlet  
(1805–1859)

The **Pigeonhole Principle** states a simple but powerful idea:

### Pigeonhole Principle

If  $n$  items are placed into  $k$  containers and  $n > k$ , then at least one container must contain more than one item.

**Generalized version:** If  $n$  items are placed into  $k$  containers, then at least one container must contain at least  $\lceil n/k \rceil$  items.

**Example 6.** In any group of 13 people, at least two must have been born in the same month.

**Solution:** There are 12 months (pigeonholes) and 13 people (pigeons). By the pigeonhole principle, at least one month must contain at least 2 people.

**Example 7.** Among any 5 points placed inside a unit square, at least two must be at distance at most  $\frac{\sqrt{2}}{2}$  from each other.

**Solution:** Divide the unit square into 4 smaller squares of side  $\frac{1}{2}$ . With 5 points and 4 squares, at least one square contains at least 2 points. The maximum distance between two points in a square of side  $\frac{1}{2}$  is the diagonal:  $\frac{\sqrt{2}}{2}$ .

**Example 8.** Prove that among any 6 integers, there exist two whose difference is divisible by 5.

**Solution:** Every integer has a remainder of 0, 1, 2, 3, or 4 when divided by 5 (these are the “pigeonholes”). With 6 integers and 5 possible remainders, at least two integers must have the same remainder. Their difference is divisible by 5.

**Example 9.** In a drawer with 10 red socks and 10 blue socks (all mixed up), how many socks must you pull out (in the dark) to guarantee a matching pair?

**Solution:** With 2 colors (pigeonholes), pulling 3 socks guarantees at least 2 of the same color.

### Quick Check

6. How many people must be in a room to guarantee that at least two share the same birthday (day of year)?
7. A drawer holds socks of 4 different colors, all mixed up. How many socks must you pull out in the dark to be sure of a matching pair?

We will see the pigeonhole principle return many times in later handouts. Bonus handout 2 (*Pigeonhole: Theme and Variations*) takes the same idea and uses it to prove monotone-subsequence theorems, lattice-midpoint results, and Dirichlet’s irrational-approximation theorem; bonus handout 6 (Ramsey theory) opens with a pigeonhole argument for the Party Theorem.

## Technique 2: Stars and Bars

The pigeonhole principle was about showing that *something must exist* — an existence tool. Our next three techniques are different: they are tools for *exact counting*. We start with stars and bars, which solves a class of problems that the basic multiplication principle alone can’t handle: distributing identical (indistinguishable) objects among distinct recipients.

### Historical Note

The visual “stars and bars” trick is folklore — nobody knows exactly who first drew it — but the underlying counting problem (splitting a whole number into a fixed number of non-negative parts) goes back to the early days of probability, when writers on games of chance such as **Abraham de Moivre** counted the ways several dice can add up to a given total. The modern stars-and-bars diagram was popularized

by **William Feller** in his landmark 1950 textbook *An Introduction to Probability Theory and Its Applications*, where it appeared as “Bose-Einstein statistics” — a model from quantum physics where identical particles are distributed among energy levels exactly the way we’re distributing identical candies among children.

How many ways can we distribute 10 identical candies among 3 children? At first this sounds like a multiplication-principle problem, but it isn’t: the candies are interchangeable, so giving Alice the “first” candy versus the “second” candy doesn’t matter — only the totals do. This type of problem—distributing identical objects into distinct groups—appears all over combinatorics, and stars and bars is the slick visual trick that handles it.

## The Stars and Bars Method

Imagine representing each candy as a star (\*) and using bars (|) to separate the portions for different children. For example, with 10 candies and 3 children:

\*\*\*\* | \*\*\* | \*\*

This represents: Child 1 gets 4 candies, Child 2 gets 3 candies, Child 3 gets 3 candies.

Another distribution:

\*\* | \*\*\*\*\* | \*

This represents: Child 1 gets 2 candies, Child 2 gets 7 candies, Child 3 gets 1 candy.

**Key insight:** Every distribution corresponds to an arrangement of 10 stars and 2 bars (we need  $k - 1$  bars to create  $k$  groups). The total number of symbols is  $10 + 2 = 12$ , and we need to choose which 2 positions are bars.

### Stars and Bars Theorem

The number of ways to distribute  $n$  identical objects into  $k$  distinct bins (where bins can be empty) is:

$$\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n}.$$

**Example 10.** How many ways can we distribute 10 identical candies among 3 children?

**Solution:** We have  $n = 10$  candies and  $k = 3$  children. Using stars and bars:

$$\binom{10 + 3 - 1}{3 - 1} = \binom{12}{2} = \frac{12 \cdot 11}{2} = 66 \text{ ways}$$

**Example 11.** How many non-negative integer solutions are there to  $x + y + z = 15$ ?

**Solution:** This is equivalent to distributing 15 identical objects into 3 bins (the values of  $x$ ,  $y$ , and  $z$ ). Using stars and bars:

$$\binom{15 + 3 - 1}{3 - 1} = \binom{17}{2} = \frac{17 \cdot 16}{2} = 136 \text{ solutions}$$

**Example 12.** How many ways can we distribute 8 identical cookies among 4 children if each child must get at least one cookie?

**Solution:** First give each child 1 cookie (using 4 cookies). Now we distribute the remaining  $8 - 4 = 4$  cookies freely among 4 children:

$$\binom{4 + 4 - 1}{4 - 1} = \binom{7}{3} = 35 \text{ ways}$$

*Alternative view:* This counts positive integer solutions to  $x + y + z + w = 8$ .

### Quick Check

8. How many ways can you distribute 7 identical balls into 4 distinct boxes?
9. How many non-negative integer solutions are there to  $a + b + c + d = 10$ ?

### Key Takeaways

- **Powers of 11:** Row  $n$  supplies the base-10 place-value coefficients of  $11^n$ ; from row 5 onward, carrying is required.
- The **multinomial coefficient**  $\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_1! \dots k_r!}$  is the coefficient of  $x_1^{k_1} \dots x_r^{k_r}$  in  $(x_1 + \dots + x_r)^n$ .
- **Parity pattern:**  $\binom{n}{k}$  is odd exactly when  $k$  fits under  $n$  in binary. Coloring the odd entries produces the Sierpinski triangle, and the all-odd rows are numbered  $2^j - 1$ .
- **Pigeonhole Principle:**  $n$  items in  $k$  boxes ( $n > k$ ) means some box has  $\geq 2$  items. Generalized: some box has  $\geq \lceil n/k \rceil$ .
- **Stars and Bars:** Distributing  $n$  identical objects into  $k$  distinct bins:  $\binom{n+k-1}{k-1}$

### Common Mistakes

- **Forgetting repeated factors:** In a multinomial coefficient, divide by the factorial of the multiplicity of *every* repeated type.
- **Forgetting to carry:**  $11^5$  is not 15101051 — the two-digit entries of row 5 must be carried.
- **Stars and bars requires identical objects:** If objects are distinct, use other methods.
- **Pigeonhole gives existence, not construction:** It proves something exists but doesn't tell you which specific items.

## Practice Problems

1. Use row 6 of Pascal's triangle to compute  $11^6$ . (Remember to carry!)
2. Compute  $101^4$  using row 4 of Pascal's triangle. Why don't you need to carry this time?
3. Is  $\binom{14}{5}$  odd or even? Use the binary-digit rule from Property 6.
4. Which entries of row 12 are odd? Use the binary-digit rule.
5. How many distinct arrangements does the word BOOKKEEPER have?
6. Find the coefficient of  $x^2yz^3$  in  $(x + y + z)^6$ .
7. Find the coefficient of  $x^5$  in  $(1 + x + x^2)^4$ . (*Hint: from each of the 4 factors you pick 1,  $x$ , or  $x^2$ . Which choices give  $x^5$ ?*)
8. How many cards must you draw from a standard 52-card deck to be sure of getting two cards of the same suit?
9. A bag holds 5 red, 5 blue, and 5 green marbles. How many must you take out, without looking, to be sure of getting 3 of the same color?
10. In a room of 8 people, some pairs shake hands (each pair shakes at most once; not every pair has to shake). Prove that no matter who shakes hands with whom, at least two people end up having shaken the same number of hands.
11. A bakery sells 4 kinds of donuts. How many ways are there to buy a box of 6 donuts? (Only the number of each kind matters.)
12. How many *positive* integer solutions does  $x + y + z = 10$  have?
13. How many non-negative integer solutions does  $x + y + z = 12$  have with  $x \geq 2$  and  $y \geq 1$ ?

## Practice Problems: Solutions

- $1 \cdot 10^6 + 6 \cdot 10^5 + 15 \cdot 10^4 + 20 \cdot 10^3 + 15 \cdot 10^2 + 6 \cdot 10 + 1 = \mathbf{1,771,561} = 11^6$ .
- $101^4 = (100 + 1)^4 = 1 \cdot 100^4 + 4 \cdot 100^3 + 6 \cdot 100^2 + 4 \cdot 100 + 1$ . Each entry gets two digits of room: 1 04 06 04 01 = **104060401**. All entries are less than 100, so nothing needs carrying.
- $14 = 1110_2$  and  $5 = 0101_2$ . In the last place, 5 has a 1 where 14 has a 0, so  $\binom{14}{5}$  is **even**. (Indeed  $\binom{14}{5} = 2002$ .)
- $12 = 1100_2$ . The numbers that fit under it are 0000, 0100, 1000, 1100, so the odd entries are  $\binom{12}{k}$  for **k=0, 4, 8, 12**. (Check:  $\binom{12}{4} = 495$  is odd, while  $\binom{12}{2} = 66$  is even.)
- 10 letters: B, O, O, K, K, E, E, E, P, R. So  $\binom{10}{1,2,2,3,1,1} = \frac{10!}{2!2!3!} = \mathbf{151200}$ .
- $\binom{6}{2,1,3} = \frac{6!}{2!1!3!} = \mathbf{60}$ .
- We need to pick  $x^2$  from  $k_2$  factors,  $x$  from  $k_1$  factors, and 1 from the rest, with  $2k_2 + k_1 = 5$  and  $k_1 + k_2 \leq 4$ . Two options work:
  - one  $x^2$ , three  $x$ 's:  $\binom{4}{1,3,0} = \frac{4!}{1!3!} = 4$  ways;
  - two  $x^2$ 's, one  $x$ , one 1:  $\binom{4}{2,1,1} = \frac{4!}{2!1!1!} = 12$  ways.
 The coefficient is  $4 + 12 = \mathbf{16}$ .
- There are 4 suits (pigeonholes). Four cards could all be different suits, but 5 cards must repeat a suit: **5**.
- With 6 marbles you could have exactly 2 of each color. The 7th marble makes some color reach 3. Answer: **7** (the generalized principle:  $\lceil 7/3 \rceil = 3$ ).
- Each person shakes hands with 0 to 7 others (8 possibilities). But if someone shakes 0 hands, no one can shake 7 hands (and vice versa). So there are really only 7 possible values for number of handshakes. With 8 people and 7 possible values, by pigeonhole, at least two people shook the same number of hands.
- 6 identical "slots" into 4 kinds:  $\binom{6+4-1}{4-1} = \binom{9}{3} = \mathbf{84}$ .
- Give each variable 1 first; then distribute the remaining 7:  $\binom{7+3-1}{2} = \binom{9}{2} = \mathbf{36}$ .
- Set  $x' = x - 2$ ,  $y' = y - 1$ . Then  $x' + y' + z = 9$  with all three non-negative:  $\binom{9+2}{2} = \binom{11}{2} = \mathbf{55}$ .

## Homework

1. Compute  $101^5$ . Then compute  $1001^3$ .
2. **Multinomial practice.** Compute:
  - (a) The coefficient of  $x^2y^3z^2$  in  $(x + y + z)^7$ .
  - (b) The number of distinct anagrams of the word COMBINATORICS.
  - (c) The coefficient of  $a^2b^2c$  in  $(2a + 3b - c)^5$ .
3. Nine different books are to be given to three students, three books each.
  - (a) In how many ways can this be done?
  - (b) Find the coefficient of  $x^3y^3z^3$  in  $(x + y + z)^9$ . Why is it the same number?
4. How many 4-digit numbers have digits that add up to 9? (For example, 1206 and 9000. Remember that the first digit can't be 0.)
5. How many ways can you distribute 20 identical candies among 4 children if each child must get at least 2 candies?
6. If 6 points are placed inside an equilateral triangle of side 2, prove that at least two points are within distance 1 of each other.
7. Six different numbers are chosen from  $1, 2, \dots, 10$ . Prove that two of them add up to 11. Is the same true for five numbers?
8. **H A sum divisible by 10.** Prove that among any 10 integers  $a_1, a_2, \dots, a_{10}$ , some of them (one or more, standing next to each other in the list) have a sum divisible by 10. (*Hint: look at the ten sums  $a_1$ ,  $a_1 + a_2$ ,  $a_1 + a_2 + a_3$ ,  $\dots$ , and their remainders when divided by 10.*)
9. **Class presentations.** A class of 20 students must each pick one weekday (Monday through Friday) to give a 10-minute presentation. The teacher only cares *how many* presentations land on each day, not who gives them. How many distinct schedules are possible?
10. **Four groups of unequal size.** A teacher wants to split a class of 19 students into four working groups of sizes 5, 5, 5, 4. The groups are not labeled — only the membership matters — but the three size-5 groups are themselves interchangeable. How many ways can this be done?
11. **Rolling  $n$  dice.** You roll  $n$  ordinary six-sided dice simultaneously.
  - (a) If the dice are distinguishable (it matters which die shows which face), how many outcomes are possible?
  - (b) If the dice are indistinguishable (you only see the multiset of faces), how many outcomes are possible?

**Pascal's Triangle** (rows 0–20; row number on the left)

|    |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
|----|---|----|-----|------|------|-------|-------|-------|--------|--------|--------|--------|--------|-------|-------|-------|------|------|-----|----|---|
| 0  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 1  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 2  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 3  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 4  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 5  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 6  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 7  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 8  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 9  |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 10 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 11 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 12 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 13 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 14 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 15 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 16 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 17 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 18 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 19 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
| 20 |   |    |     |      |      |       |       |       |        |        |        |        |        |       |       |       |      |      |     |    |   |
|    | 1 | 20 | 190 | 1140 | 4845 | 15504 | 38760 | 77520 | 125970 | 167960 | 184756 | 167960 | 125970 | 77520 | 38760 | 15504 | 4845 | 1140 | 190 | 20 | 1 |

**Pascal's Triangle** (rows 0–20; row number on the left)

|    |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       |        |       |        |       |        |       |        |       |        |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
|----|--|---|---|----|----|-----|-----|------|-----|------|------|-------|-------|-------|-------|-------|-------|--------|-------|--------|-------|--------|-------|--------|-------|--------|-------|-------|-------|-------|-------|-------|------|------|-----|------|-----|-----|----|----|---|---|--|
| 0  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       |        |       |        |       | 1      |       |        |       |        |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 1  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       |        |       |        | 1     |        | 1     |        |       |        |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 2  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       |        |       | 1      |       | 2      |       | 1      |       |        |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 3  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       |        | 1     |        | 3     |        | 3     |        | 1     |        |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 4  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       |       | 1      |       | 4      |       | 6      |       | 4      |       | 1      |       |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 5  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       |       | 1     |        | 5     |        | 10    |        | 10    |        | 5     |        | 1     |       |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 6  |  |   |   |    |    |     |     |      |     |      |      |       |       |       |       | 1     |       | 6      |       | 15     |       | 20     |       | 15     |       | 6      |       | 1     |       |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 7  |  |   |   |    |    |     |     |      |     |      |      |       |       |       | 1     |       | 7     |        | 21    |        | 35    |        | 35    |        | 21    |        | 7     |       | 1     |       |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 8  |  |   |   |    |    |     |     |      |     |      |      |       |       | 1     |       | 8     |       | 28     |       | 56     |       | 70     |       | 56     |       | 28     |       | 8     |       | 1     |       |       |      |      |     |      |     |     |    |    |   |   |  |
| 9  |  |   |   |    |    |     |     |      |     |      |      |       | 1     |       | 9     |       | 36    |        | 84    |        | 126   |        | 126   |        | 84    |        | 36    |       | 9     |       | 1     |       |      |      |     |      |     |     |    |    |   |   |  |
| 10 |  |   |   |    |    |     |     |      |     |      |      | 1     |       | 10    |       | 45    |       | 120    |       | 210    |       | 252    |       | 210    |       | 120    |       | 45    |       | 10    |       | 1     |      |      |     |      |     |     |    |    |   |   |  |
| 11 |  |   |   |    |    |     |     |      |     |      | 1    |       | 11    |       | 55    |       | 165   |        | 330   |        | 462   |        | 462   |        | 330   |        | 165   |       | 55    |       | 11    |       | 1    |      |     |      |     |     |    |    |   |   |  |
| 12 |  |   |   |    |    |     |     |      |     | 1    |      | 12    |       | 66    |       | 220   |       | 495    |       | 792    |       | 924    |       | 792    |       | 495    |       | 220   |       | 66    |       | 12    |      | 1    |     |      |     |     |    |    |   |   |  |
| 13 |  |   |   |    |    |     |     |      | 1   |      | 13   |       | 78    |       | 286   |       | 715   |        | 1287  |        | 1716  |        | 1716  |        | 1287  |        | 715   |       | 286   |       | 78    |       | 13   |      | 1   |      |     |     |    |    |   |   |  |
| 14 |  |   |   |    |    |     |     | 1    |     | 14   |      | 91    |       | 364   |       | 1001  |       | 2002   |       | 3003   |       | 3432   |       | 3003   |       | 2002   |       | 1001  |       | 364   |       | 91    |      | 14   |     | 1    |     |     |    |    |   |   |  |
| 15 |  |   |   |    |    |     | 1   |      | 15  |      | 105  |       | 455   |       | 1365  |       | 3003  |        | 5005  |        | 6435  |        | 6435  |        | 5005  |        | 3003  |       | 1365  |       | 455   |       | 105  |      | 15  |      | 1   |     |    |    |   |   |  |
| 16 |  |   |   |    |    | 1   |     | 16   |     | 120  |      | 560   |       | 1820  |       | 4368  |       | 8008   |       | 11440  |       | 12870  |       | 11440  |       | 8008   |       | 4368  |       | 1820  |       | 560   |      | 120  |     | 16   |     | 1   |    |    |   |   |  |
| 17 |  |   |   |    | 1  |     | 17  |      | 136 |      | 680  |       | 2380  |       | 6188  |       | 12376 |        | 19448 |        | 24310 |        | 24310 |        | 19448 |        | 12376 |       | 6188  |       | 2380  |       | 680  |      | 136 |      | 17  |     | 1  |    |   |   |  |
| 18 |  |   |   | 1  |    | 18  |     | 153  |     | 816  |      | 3060  |       | 8568  |       | 18564 |       | 31824  |       | 43758  |       | 48620  |       | 43758  |       | 31824  |       | 18564 |       | 8568  |       | 3060  |      | 816  |     | 153  |     | 18  |    | 1  |   |   |  |
| 19 |  |   | 1 |    | 19 |     | 171 |      | 969 |      | 3876 |       | 11628 |       | 27132 |       | 50388 |        | 75582 |        | 92378 |        | 92378 |        | 75582 |        | 50388 |       | 27132 |       | 11628 |       | 3876 |      | 969 |      | 171 |     | 19 |    | 1 |   |  |
| 20 |  | 1 |   | 20 |    | 190 |     | 1140 |     | 4845 |      | 15504 |       | 38760 |       | 77520 |       | 125970 |       | 167960 |       | 184756 |       | 167960 |       | 125970 |       | 77520 |       | 38760 |       | 15504 |      | 4845 |     | 1140 |     | 190 |    | 20 |   | 1 |  |

## Quick Check Answers

1. Row 5 contains the two-digit entries 10, so reading the row as digits ("15101051) is wrong. Carrying gives  $1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10 + 1 = 161051$ .
2. Rows  $15 = 2^4 - 1$  and  $31 = 2^5 - 1$  are entirely odd; row 16 is not (for example,  $\binom{16}{1} = 16$  is even).
3.  $6 = 110_2$  and  $3 = 011_2$ . In the last place, 3 has a 1 where 6 has a 0, so  $\binom{6}{3}$  is **even**. Indeed  $\binom{6}{3} = 20$ .
4.  $\binom{6}{3,2,1} = \frac{6!}{3!2!1!} = 60$  (three A's, two N's, one B).
5.  $\binom{4}{2,1,1} = \frac{4!}{2!1!1!} = 12$ .
6. There are 366 possible birthdays (pigeonholes), counting February 29. So 367 people guarantee that two share a birthday.
7. **5 socks**: four socks could all be different colors, but the fifth must repeat one of the 4 colors.
8.  $\binom{7+4-1}{4-1} = \binom{10}{3} = 120$  ways
9.  $\binom{10+4-1}{4-1} = \binom{13}{3} = 286$  solutions