

MATH 8

PROPERTIES OF PASCAL'S TRIANGLE — SHORT 2026–27

Stare at Pascal's triangle long enough and patterns start jumping out: every row reads the same backwards, the row sums double, and diagonals add up to single entries. In this handout we prove four of these patterns. Our favorite tool is **double counting**: count the same collection in two different ways — the two answers must be equal.

Property 1: Symmetry

$$\binom{n}{k} = \binom{n}{n-k}.$$

Why? Choosing the k people who go on a trip is the same as choosing the $n - k$ who stay home. Every choice of one group is a choice of the other. That's why each row of the triangle is a palindrome: 1, 5, 10, 10, 5, 1.

Property 2: Row Sums

Rows 0, 1, 2, 3, 4 add up to 1, 2, 4, 8, 16: the sums double every time. In general,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Counting reason. $\binom{n}{k}$ counts the k -element subsets of an n -element set, so the left side counts *all* subsets, sorted by size. But we can also build a subset by deciding for each element “in or out”: $2 \cdot 2 \cdots 2 = 2^n$ ways. Same collection, so the answers are equal.

Construction reason. Each entry of row n helps build *two* entries of row $n + 1$ (the one below-left and the one below-right). So when we add up row $n + 1$, every number of row n is counted twice:

$$\underbrace{(0+1)}_1 + \underbrace{(1+3)}_4 + \underbrace{(3+3)}_6 + \underbrace{(3+1)}_4 + \underbrace{(1+0)}_1 = 2(1+3+3+1) = 16.$$

Each row's sum is double the one above it, starting from 1: 1, 2, 4, 8, 16, ...

Property 3: Alternating Row Sums

What if we alternate the signs? For every $n \geq 1$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

Watch row 4 being built from row 3, and give its entries alternating signs:

$$1 - 4 + 6 - 4 + 1 = (0 + 1) - (1 + 3) + (3 + 3) - (3 + 1) + (1 + 0) = 0.$$

Each number of row 3 goes down to two *neighboring* places, and neighbors get opposite signs — so everything cancels. Only row 0 (the lone 1) isn't built from a row above it, and that is exactly the exception.

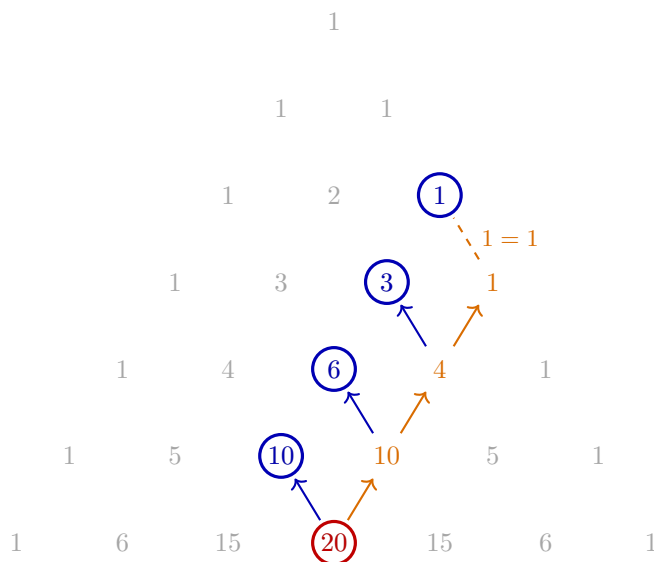
In words: a set with at least one element has as many *even-sized* subsets as *odd-sized* ones. Can you see why directly? You'll find out in the homework.

Property 4: The Hockey Stick

Add the entries down a diagonal, and the total is the entry just below and to the side — the picture looks like a hockey stick:

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

Unzipping the blade. Here $20 = 1 + 3 + 6 + 10$. Split the blade into the two numbers above it, keep the one on the handle, and keep splitting the leftover:



$$20 = 10 + 10 = 10 + 6 + 4 = 10 + 6 + 3 + 1 = 10 + 6 + 3 + 1.$$

The last leftover is a 1 on the edge — the same as the 1 at the top of the handle. This works for every hockey stick.

Counting reason. $\binom{6}{3} = 20$ counts the 3-element subsets of $\{1, \dots, 6\}$. Sort them by their largest element (3, 4, 5, or 6): the other two elements come from $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, or $\{1, \dots, 5\}$ — that's $1 + 3 + 6 + 10$ subsets. The handle is the list of these groups!

Common Mistakes

- **Row sum** is 2^n , not $n!$ or n^2 .
- **Alternating sums** need $n \geq 1$: row 0 gives 1, not 0.
- **Hockey stick direction**: the handle runs down a diagonal with the *same bottom number* r ; the blade is one row lower and one step further in.

Homework

1. Use symmetry to compute $\binom{50}{47}$.
2. Find all k for which $\binom{20}{k} = \binom{20}{3k-4}$.
3. Find $\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \cdots + \binom{11}{10}$.
4. **Discover it yourself: even and odd subsets.** Property 3 says that for $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

that is, a set with $n \geq 1$ elements has as many *even-sized* subsets as *odd-sized* ones. We proved this with algebra. In this problem you will find a reason you can *see*.

- (a) List all 8 subsets of $\{a, b, c\}$ in two columns: even size and odd size. (The empty set has size 0, which is even.)
 - (b) Here is a move called *toggling a* : if a is in the subset, take it out; if it is not, put it in. Toggle a in each of your 8 subsets. Does an even-sized subset become even or odd? Why must this always happen?
 - (c) Draw a line from each even-sized subset to the subset you get by toggling a . What happens if you toggle a twice? Explain why every odd-sized subset ends up with exactly one line.
 - (d) Now think about $\{a, b, c, d\}$, which has 16 subsets. How many are even-sized? Do you need to list them all to be sure?
 - (e) Explain why the same toggling argument works for *any* set with at least one element, and conclude Property 3.
 - (f) Where does the argument break down for $n = 0$? Does that match the exception in Property 3?
5. Use the hockey stick identity to compute $\binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2}$.
 6. **M** Use the hockey stick identity to compute $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 9 \cdot 10$. (*Hint: $k(k+1) = 2\binom{k+1}{2}$.*)
 7. **M** **Committees of any size with a chair.** From n people we choose a committee of any size (at least one person) and name one of its members the chair. Prove the identity below by counting.
 - (a) Count the possibilities by first choosing the size k of the committee.
 - (b) Count them again by first choosing the chair.
 - (c) Conclude that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$.
 8. **Investigation: odd numbers in Pascal's triangle.** Use the Pascal's triangle page at the end of this homework and color every *odd* entry. (You only need the last digit to tell whether a number is odd.)
 - (a) Describe the picture you get.
 - (b) Some rows are colored completely. Which ones? Predict the next completely colored row after row 20.
 - (c) Rows 0–3 form a small colored triangle. Find copies of it in rows 4–7. How many are there, and what is between them?
 - (d) Now think of rows 0–7 as one bigger triangle. Find its copies in rows 8–15. What do you predict for rows 16–31?
 - (e) Count the colored entries in each row from 0 to 20. What kind of numbers do you get? Which rows have only two colored entries?

(Later we will see why all of this happens.)

9. Investigation: powers of 11.

- (a) Compute 11^2 , 11^3 , and 11^4 . Compare them with rows 2, 3, and 4 of Pascal's triangle. What do you notice?
- (b) Explain the pattern: write $11 = 10 + 1$ and expand $(10 + 1)^4$ with the binomial theorem.
- (c) Now $11^5 = 161051$. Why can't you just read it off row 5 (1, 5, 10, 10, 5, 1)? Show how row 5 still gives 161051 if you carry.
- (d) Without multiplying, find the last three digits of 11^{10} . (*Hint: which terms of $(10 + 1)^{10}$ can change the last three digits?*)

Pascal's Triangle (rows 0–20; row number on the left)

0												1																															
1											1		1																														
2										1		2		1																													
3									1		3		3		1																												
4								1		4		6		4		1																											
5							1		5		10		10		5		1																										
6						1		6		15		20		15		6		1																									
7					1		7		21		35		35		21		7		1																								
8				1		8		28		56		70		56		28		8		1																							
9			1		9		36		84		126		126		84		36		9		1																						
10		1		10		45		120		210		252		210		120		45		10		1																					
11		1		11		55		165		330		462		462		330		165		55		11		1																			
12		1		12		66		220		495		792		924		792		495		220		66		12		1																	
13		1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1															
14		1		14		91		364		1001		2002		3003		3432		3003		2002		1001		364		91		14		1													
15		1		15		105		455		1365		3003		5005		6435		6435		5005		3003		1365		455		105		15		1											
16		1		16		120		560		1820		4368		8008		11440		12870		11440		8008		4368		1820		560		120		16		1									
17		1		17		136		680		2380		6188		12376		19448		24310		24310		19448		12376		6188		2380		680		136		17		1							
18		1		18		153		816		3060		8568		18564		31824		43758		48620		43758		31824		18564		8568		3060		816		153		18		1					
19		1		19		171		969		3876		11628		27132		50388		75582		92378		92378		75582		50388		27132		11628		3876		969		171		19		1			
20		1		20		190		1140		4845		15504		38760		77520		125970		167960		184756		167960		125970		77520		38760		15504		4845		1140		190		20		1	

Pascal's Triangle (rows 0–20; row number on the left)

0												1																															
1											1		1																														
2										1		2		1																													
3									1		3		3		1																												
4								1		4		6		4		1																											
5							1		5		10		10		5		1																										
6						1		6		15		20		15		6		1																									
7					1		7		21		35		35		21		7		1																								
8				1		8		28		56		70		56		28		8		1																							
9			1		9		36		84		126		126		84		36		9		1																						
10		1		10		45		120		210		252		210		120		45		10		1																					
11		1		11		55		165		330		462		462		330		165		55		11		1																			
12		1		12		66		220		495		792		924		792		495		220		66		12		1																	
13		1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1															
14		1		14		91		364		1001		2002		3003		3432		3003		2002		1001		364		91		14		1													
15		1		15		105		455		1365		3003		5005		6435		6435		5005		3003		1365		455		105		15		1											
16		1		16		120		560		1820		4368		8008		11440		12870		11440		8008		4368		1820		560		120		16		1									
17		1		17		136		680		2380		6188		12376		19448		24310		24310		19448		12376		6188		2380		680		136		17		1							
18		1		18		153		816		3060		8568		18564		31824		43758		48620		43758		31824		18564		8568		3060		816		153		18		1					
19		1		19		171		969		3876		11628		27132		50388		75582		92378		92378		75582		50388		27132		11628		3876		969		171		19		1			
20		1		20		190		1140		4845		15504		38760		77520		125970		167960		184756		167960		125970		77520		38760		15504		4845		1140		190		20		1	