

MATH 8

PROPERTIES OF PASCAL'S TRIANGLE — FULL 2026–27

Introduction

Look at Pascal's triangle long enough and patterns start jumping out: every row is a palindrome, the row sums double, certain diagonals add up to single entries, and the odd numbers form a fractal. In this handout we prove four of these properties — symmetry, row sums, alternating row sums, and the hockey stick. Two more surprises, the fractal and a strange link with the powers of 11, you will investigate yourself in the homework.

For most properties we give two proofs — one by algebra, one by *counting something in two ways*. The algebra checks that an identity is true; the counting argument shows *why*.

Row 0					1												
Row 1					1		1										
Row 2					1		2		1								
Row 3					1		3		3		1						
Row 4					1		4		6		4		1				
Row 5					1		5		10		10		5		1		
Row 6					1		6		15		20		15		6		1

Method: Double Counting

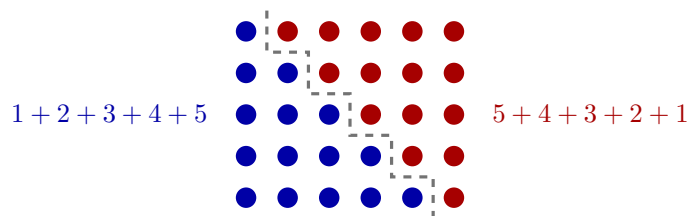
Before we dive into the properties, let's name a proof technique we are about to use repeatedly. It is called **double counting** (or *counting in two ways*), and along with bijective proof (Handout 2), it is one of the great combinatorial methods.

Double counting

To prove that two expressions are equal, find a single collection of objects, and *count it two different ways*. If both counts are correct, the two expressions must be equal.

The technique sounds almost too simple to be useful, but it is responsible for some of the cleanest proofs in mathematics. Here is the prototype:

Example 1 ($1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, by double counting). Consider a rectangular grid of dots with n rows and $n + 1$ columns — a total of $n(n + 1)$ dots. Cut this rectangle along a staircase into two halves of the same shape. The upper-left staircase has 1 dot in its top row, 2 in the next, then 3, all the way down to n dots in its bottom row, for a total of $1 + 2 + \cdots + n$ dots. The lower-right staircase, by symmetry, also has $1 + 2 + \cdots + n$ dots.



(the case $n = 5$: total $n(n+1) = 5 \times 6 = 30$ dots)

Counting the same rectangle two ways: $n(n+1) = 2(1+2+\cdots+n)$, so

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

(This is Gauss's schoolboy trick, recast as double counting.)

We will see double counting in action throughout this handout:

- Property 2 (Row sum = 2^n): count subsets of $\{1, \dots, n\}$ by size vs. by "each element in or out."
- Property 4 (Hockey stick): count subsets of fixed size by tagging their largest element.

The argument has the same shape every time — count one set in two ways, and read off the identity.

Property 1: Symmetry

We start with the property that's visible at a glance. Look at any row of Pascal's triangle: 1, 5, 10, 10, 5, 1 or 1, 6, 15, 20, 15, 6, 1. The rows are palindromes — they read the same left to right as right to left. In symbols:

Symmetry of Pascal's Triangle

$$\binom{n}{k} = \binom{n}{n-k}$$

This says that each row of Pascal's Triangle is symmetric — it reads the same forwards and backwards.

Proof 1: Combinatorial

$\binom{n}{k}$ counts the number of ways to choose k items from n items.

$\binom{n}{n-k}$ counts the number of ways to choose $n-k$ items from n items.

But choosing which k items to *take* is the same as choosing which $n-k$ items to *leave behind*. Every selection of k items determines a selection of $n-k$ items (the rest), and vice versa. Therefore $\binom{n}{k} = \binom{n}{n-k}$. $\square$

Proof 2: Algebraic

Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

The formula is symmetric in k and $n-k$. $\square$

Proof 3: Using the Binomial Theorem

Since $(a + b)^n = (b + a)^n$, the two expansions are the same polynomial. In $(a + b)^n$ the term $a^{n-k}b^k$ has coefficient $\binom{n}{k}$. In $(b + a)^n$ the same term $b^k a^{n-k}$ is the one where a appears $n - k$ times, so its coefficient is $\binom{n}{n-k}$. Equal polynomials have equal coefficients: $\binom{n}{k} = \binom{n}{n-k}$. $\square$

Quick Check

1. Verify that $\binom{7}{2} = \binom{7}{5}$ using the formula.
2. Give a combinatorial explanation for why choosing 2 items from 7 is equivalent to choosing 5 items from 7.

Property 2: Row Sums

If you add up the entries in each row of Pascal's triangle, what do you get? Row 0 gives 1; row 1 gives 2; row 2 gives 4; row 3 gives 8. The sums double every row — they're powers of 2.

Sum of Row n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Proof 1: Using the Binomial Theorem

Set $a = 1$ and $b = 1$ in the binomial theorem:

$$\begin{aligned} (1 + 1)^n &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1} \cdot 1 + \cdots + \binom{n}{n}1^n \\ 2^n &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \end{aligned}$$

$\square$

Proof 2: Combinatorial

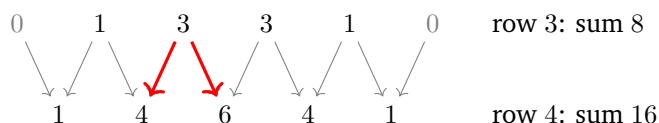
The left side $\sum_{k=0}^n \binom{n}{k}$ counts the number of subsets of an n -element set (summing over all possible subset sizes).

The right side 2^n also counts subsets: for each of the n elements, we have 2 choices (include it or not), giving 2^n total subsets.

Since both count the same thing, they are equal. $\square$

Proof 3: Straight from the Construction

Recall how we build row $n + 1$ from row n : each new entry is the sum of the two entries just above it. (Put an invisible 0 beyond each end of row n ; then the 1s at the edges follow the same rule.) Let's watch where the numbers of row 3 go when we build row 4:



Every number in row 3 sends a copy of itself to *two* places in row 4: one down to the left and one down to the right. (The red arrows follow the first 3: it helps make both the 4 and the 6.) So when we add up row 4, every number of row 3 is counted exactly twice:

$$\underbrace{(0+1)}_1 + \underbrace{(1+3)}_4 + \underbrace{(3+3)}_6 + \underbrace{(3+1)}_4 + \underbrace{(1+0)}_1 = 2(1+3+3+1) = 16.$$

Nothing here was special about row 3: in every row, each entry is used twice to build the next row. Therefore

$$(\text{sum of row } n+1) = 2 \times (\text{sum of row } n).$$

Row 0 is just 1, and each new row doubles the sum: 1, 2, 4, 8, 16, ... So the sum of row n is 2^n . $\square$

Quick Check

3. What is the sum of all numbers in row 10 of Pascal's Triangle?

4. A set has 8 elements. How many subsets does it have?

Property 3: Alternating Row Sums

We just saw what happens when you *add* the entries of a row. What if you alternate the signs — adding, then subtracting, then adding again? Surprisingly, the answer is always zero (for any positive n).

Alternating Sum of Row n (for $n \geq 1$)

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Proof 1: Using the Binomial Theorem

Set $a = 1$ and $b = -1$:

$$(1 + (-1))^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1) + \binom{n}{2}1^{n-2}(-1)^2 + \cdots$$

$$0^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots$$

For $n \geq 1$, we have $0^n = 0$. $\square$

Proof 2: Straight from the Construction

Look again at the picture from Property 2, where row 4 is built from row 3. Now give the entries of row 4 alternating signs, and write each one as the sum of the two numbers above it:

$$1 - 4 + 6 - 4 + 1 = (0 + 1) - (1 + 3) + (3 + 3) - (3 + 1) + (1 + 0).$$

Open the brackets:

$$0 + \underbrace{1-1}_0 - \underbrace{3+3}_0 + \underbrace{3-3}_0 - \underbrace{1+1}_0 + 0 = 0.$$

Each number of row 3 goes down to *two neighboring* places in row 4 — and neighbors always get *opposite* signs. So every number of row 3 appears once with a plus and once with a minus, and it cancels itself out.

Nothing here depends on what the numbers in row 3 are: the same cancellation happens whenever a row is built from the row above it. Every row from row 1 on is built this way, so every one of them has alternating sum 0. Only row 0, the lone 1 at the top, is not built from anything — and that is exactly why it is the one exception. $\square$

What does it count? Since $\binom{n}{k}$ counts the k -element subsets of an n -element set, Property 3 says something surprising: a set with at least one element has exactly as many *even-sized* subsets as *odd-sized* ones. Is there a reason you can see directly, without any algebra? You will discover one yourself in the homework (“Discover it yourself: even and odd subsets”).

Quick Check

5. Verify that $1 - 5 + 10 - 10 + 5 - 1 = 0$ (row 5).
6. How many even-sized subsets does a 6-element set have?

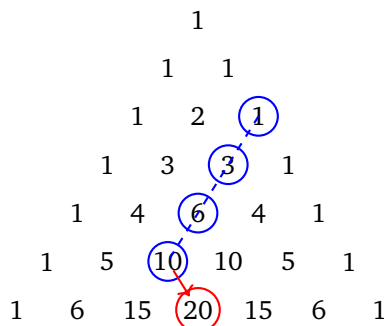
Property 4: The Hockey Stick Identity

The first three properties involved single rows of Pascal’s triangle. The Hockey Stick Identity is our first identity that runs *along a diagonal*. If you slide down one of the triangle’s diagonals, summing entries as you go, the total turns out to be the single entry just below-and-one-step-over from where you stopped.

Hockey Stick Identity

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

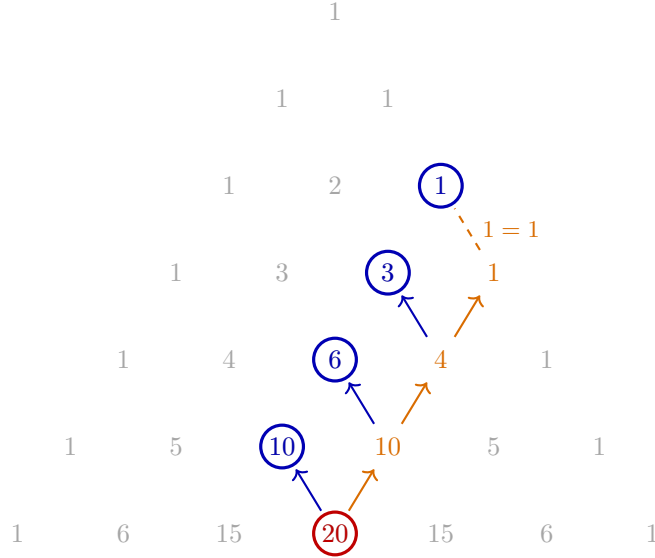
This identity gets its name from the shape formed when you highlight these entries in Pascal’s Triangle — it looks like a hockey stick!



Example: $1 + 3 + 6 + 10 = 20$, i.e., $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}$.
The four blue circles form the “handle”; the red circle is the “blade.”

Proof 1: Unzipping the Blade

Every entry of Pascal’s triangle is the sum of the two entries just above it. Let’s use this rule over and over, starting from the blade **20**, and watch what happens.



- The blade splits into the two numbers above it: $20 = 10 + 10$. The left 10 is on the handle — keep it. The right 10 is the “leftover.”
- Split the leftover the same way: $10 = 6 + 4$. Again one part lands on the handle and the other is a new, smaller leftover.
- Again: $4 = 3 + 1$.
- The last leftover 1 sits on the edge of the triangle, so it is equal to the 1 at the top of the handle (every edge entry is 1).

Putting it together:

$$20 = 10 + 10 = 10 + 6 + 4 = 10 + 6 + 3 + 1 = 10 + 6 + 3 + 1.$$

Nothing here depended on the particular numbers. For *any* hockey stick, unzip the blade: each split puts one number on the handle and passes a leftover one step up the neighboring diagonal, until the leftover reaches the edge of the triangle, where it is a 1 — exactly like the top of the handle. So the blade always equals the sum of the handle. $\square$

Proof 2: Counting Subsets by Their Largest Element

Here is a counting reason for the same example. The blade $\binom{6}{3} = 20$ counts the 3-element subsets of $\{1, 2, 3, 4, 5, 6\}$. Sort them by their *largest* element:

largest element	the other two come from	number of subsets
3	$\{1, 2\}$	$\binom{2}{2} = 1$
4	$\{1, 2, 3\}$	$\binom{3}{2} = 3$
5	$\{1, 2, 3, 4\}$	$\binom{4}{2} = 6$
6	$\{1, 2, 3, 4, 5\}$	$\binom{5}{2} = 10$

Every subset has exactly one largest element, so $1 + 3 + 6 + 10 = 20$. The handle is the list of these groups!

The same sorting works in general: an $(r + 1)$ -element subset of $\{1, \dots, n + 1\}$ whose largest element is m is completed by choosing the other r elements from $\{1, \dots, m - 1\}$, in $\binom{m-1}{r}$ ways. Adding over $m = r + 1, \dots, n + 1$ gives the handle, $\binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n}{r}$, and the total is the blade, $\binom{n+1}{r+1}$. $\square$

Example 2. Compute $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$ using the hockey stick identity.

Solution: Note that $k = \binom{k}{1}$. So:

$$1 + 2 + \cdots + 10 = \binom{1}{1} + \binom{2}{1} + \cdots + \binom{10}{1} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$$

Quick Check

7. Compute $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}$ using the hockey stick identity.

Key Takeaways

- **Symmetry:** $\binom{n}{k} = \binom{n}{n-k}$ (choosing k items is equivalent to leaving $n - k$ items)
- **Row sum** (the total number of subsets of an n -set):

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

$$\text{in sigma notation: } \sum_{k=0}^n \binom{n}{k} = 2^n$$

- **Alternating sum**, for $n \geq 1$ (even-sized and odd-sized subsets are equally numerous):

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

$$\text{in sigma notation: } \sum_{k=0}^n (-1)^k \binom{n}{k} = 0$$

- **Hockey stick** (the handle adds up to the blade):

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

$$\text{in sigma notation: } \sum_{i=r}^n \binom{i}{r} = \binom{n+1}{r+1}$$

- **Method:** *Double counting* — count one collection two ways. Used for the row sum and the hockey stick above.

Common Mistakes

- **Forgetting that row sum is 2^n , not $n!$:** The entries in row n involve factorials, but their sum is simply 2^n .
- **Applying hockey stick in wrong direction:** The diagonal must go “down” (increasing n) with fixed second index r .

Practice Problems

1. Use symmetry to compute $\binom{100}{98}$ without a calculator.
2. Find n if $\binom{n}{4} = \binom{n}{9}$.
3. Find the sum and the alternating sum of the entries in row 7.
4. Compute $\binom{8}{0} - \binom{8}{1} + \binom{8}{2} - \cdots + \binom{8}{8}$ and $\binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7}$ without computing the entries.
5. How many subsets of a 10-element set have an even number of elements?
6. How many subsets of a 10-element set have at least 2 elements?
7. Prove that for any integer $n \geq 1$, $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$.
(Hint: What happens when you add the row sum and the alternating row sum? When you subtract them?)
8. A committee of any size (but at least 1 person) must be chosen from a group of 10 people. How many ways can this be done?
9. You have one coin of each kind: 1¢, 5¢, 10¢, 25¢, and 50¢. How many different amounts of money can you pay using at least one coin?
10. Which row of Pascal's triangle has entries that add up to 512?
11. Without computing the entries, find $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4}$.
12. How many subsets of $\{1, 2, \dots, 8\}$ contain the number 1?
13. Use the hockey stick identity to find:
 - (a) $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4}$
 - (b) $1 + 3 + 6 + 10 + 15 + 21$ (triangular numbers)
14. Use the hockey stick identity to compute $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{10}{2}$.
15. **The hockey stick the other way.** Compute $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3}$. (Hint: use symmetry to rewrite each term with the same bottom number.)
16. Find n if $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = 84$.
17. How many 3-element subsets of $\{1, 2, \dots, 12\}$ have largest element 8?
18. Prove that for any positive integer n :

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$$

(Hint: Use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$ and then recognize a row sum.)

Practice Problems: Solutions

- $\binom{100}{98} = \binom{100}{2} = \frac{100 \cdot 99}{2} = \mathbf{4950}$.
- Two different entries in the same row are equal only when they are mirror images, so $4 + 9 = n$: **n=13**.
- Sum: $2^7 = \mathbf{128}$. Alternating sum: **0** (Property 3, since $7 \geq 1$).
- The alternating sum is **0** (Property 3). So the odd-position entries add up to half the row sum: $2^8/2 = \mathbf{128}$.
- Even-sized and odd-sized subsets are equally numerous, so half of all 2^{10} subsets are even-sized: **512**.
- All subsets minus the empty set minus the 10 one-element subsets: $1024 - 1 - 10 = \mathbf{1013}$.
- Let $S = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ (row sum).
Let $A = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots = 0$ (alternating sum).
Let E = (sum of even-positioned terms) and O = (sum of odd-positioned terms).
Then $S = E + O = 2^n$ and $A = E - O = 0$.
Adding: $2E = 2^n$, so $E = 2^{n-1}$.
Subtracting: $2O = 2^n$, so $O = 2^{n-1}$.
- We want non-empty subsets of a 10-element set. Total subsets = $2^{10} = 1024$. Excluding the empty set: $1024 - 1 = 1023$ ways.
- Each amount comes from a non-empty set of coins: $2^5 - 1 = 31$ sets. Different sets give different amounts here (each coin is worth more than all the smaller ones together: $5 > 1$, $10 > 1 + 5$, $25 > 1 + 5 + 10$, $50 > 1 + 5 + 10 + 25$), so there are **31** amounts.
- Row sums are powers of 2, and $512 = 2^9$: **row 9**.
- Row 9 has 10 entries, and by symmetry its left half $\binom{9}{0}, \dots, \binom{9}{4}$ and its right half $\binom{9}{5}, \dots, \binom{9}{9}$ have the same sum. The whole row adds up to $2^9 = 512$, so the answer is **256**.
- Such a subset is $\{1\}$ together with *any* subset of $\{2, \dots, 8\}$: $2^7 = \mathbf{128}$ — exactly half of all 256 subsets.
- (a) By hockey stick: $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4} = \binom{9}{5} = 126$
(b) The triangular numbers are $\binom{k+1}{2}$: $1+3+6+10+15+21 = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} = \binom{8}{3} = 56$
- $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{10}{2} = \binom{11}{3} = \mathbf{165}$.
- By symmetry the sum is $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}$, which the hockey stick identity turns into $\binom{7}{4} = \mathbf{35}$.
Check: $1 + 4 + 10 + 20 = 35$. (So hockey sticks also run along the other diagonals.)
- By the hockey stick identity the sum is $\binom{n+1}{3}$. Since $\binom{9}{3} = 84$, we get $n + 1 = 9$, so **n=8**.
- The other two elements are chosen from $\{1, \dots, 7\}$: $\binom{7}{2} = \mathbf{21}$.
- Use $k\binom{n}{k} = n\binom{n-1}{k-1}$:

$$\sum_{k=1}^n k\binom{n}{k} = \sum_{k=1}^n n\binom{n-1}{k-1} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = n \cdot 2^{n-1}$$

Homework

1. Use symmetry to compute $\binom{50}{47}$.
2. Find all k for which $\binom{20}{k} = \binom{20}{3k-4}$.
3. Find $\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \cdots + \binom{11}{10}$.
4. **Discover it yourself: even and odd subsets.** Property 3 says that for $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

that is, a set with $n \geq 1$ elements has as many *even-sized* subsets as *odd-sized* ones. We proved this with algebra. In this problem you will find a reason you can *see*.

- (a) List all 8 subsets of $\{a, b, c\}$ in two columns: even size and odd size. (The empty set has size 0, which is even.)
 - (b) Here is a move called *toggling a* : if a is in the subset, take it out; if it is not, put it in. Toggle a in each of your 8 subsets. Does an even-sized subset become even or odd? Why must this always happen?
 - (c) Draw a line from each even-sized subset to the subset you get by toggling a . What happens if you toggle a twice? Explain why every odd-sized subset ends up with exactly one line.
 - (d) Now think about $\{a, b, c, d\}$, which has 16 subsets. How many are even-sized? Do you need to list them all to be sure?
 - (e) Explain why the same toggling argument works for *any* set with at least one element, and conclude Property 3.
 - (f) Where does the argument break down for $n = 0$? Does that match the exception in Property 3?
5. Use the hockey stick identity to compute $\binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2}$.
 6. **M** Use the hockey stick identity to compute $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 9 \cdot 10$. (Hint: $k(k+1) = 2\binom{k+1}{2}$.)
 7. **M** **Committees of any size with a chair.** From n people we choose a committee of any size (at least one person) and name one of its members the chair. Prove the identity below by counting.
 - (a) Count the possibilities by first choosing the size k of the committee.
 - (b) Count them again by first choosing the chair.
 - (c) Conclude that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$.
 8. **Investigation: odd numbers in Pascal's triangle.** Use the Pascal's triangle page at the end of this homework and color every *odd* entry. (You only need the last digit to tell whether a number is odd.)
 - (a) Describe the picture you get.
 - (b) Some rows are colored completely. Which ones? Predict the next completely colored row after row 20.
 - (c) Rows 0–3 form a small colored triangle. Find copies of it in rows 4–7. How many are there, and what is between them?
 - (d) Now think of rows 0–7 as one bigger triangle. Find its copies in rows 8–15. What do you predict for rows 16–31?
 - (e) Count the colored entries in each row from 0 to 20. What kind of numbers do you get? Which rows have only two colored entries?

(Later we will see why all of this happens.)

9. Investigation: powers of 11.

- (a) Compute 11^2 , 11^3 , and 11^4 . Compare them with rows 2, 3, and 4 of Pascal's triangle. What do you notice?
- (b) Explain the pattern: write $11 = 10 + 1$ and expand $(10 + 1)^4$ with the binomial theorem.
- (c) Now $11^5 = 161051$. Why can't you just read it off row 5 (1, 5, 10, 10, 5, 1)? Show how row 5 still gives 161051 if you carry.
- (d) Without multiplying, find the last three digits of 11^{10} . (*Hint: which terms of $(10 + 1)^{10}$ can change the last three digits?*)

Pascal's Triangle (rows 0–20; row number on the left)

0												1																															
1											1		1																														
2										1		2		1																													
3									1		3		3		1																												
4								1		4		6		4		1																											
5							1		5		10		10		5		1																										
6						1		6		15		20		15		6		1																									
7					1		7		21		35		35		21		7		1																								
8				1		8		28		56		70		56		28		8		1																							
9			1		9		36		84		126		126		84		36		9		1																						
10		1		10		45		120		210		252		210		120		45		10		1																					
11		1		11		55		165		330		462		462		330		165		55		11		1																			
12		1		12		66		220		495		792		924		792		495		220		66		12		1																	
13		1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1															
14		1		14		91		364		1001		2002		3003		3432		3003		2002		1001		364		91		14		1													
15		1		15		105		455		1365		3003		5005		6435		6435		5005		3003		1365		455		105		15		1											
16		1		16		120		560		1820		4368		8008		11440		12870		11440		8008		4368		1820		560		120		16		1									
17		1		17		136		680		2380		6188		12376		19448		24310		24310		19448		12376		6188		2380		680		136		17		1							
18		1		18		153		816		3060		8568		18564		31824		43758		48620		43758		31824		18564		8568		3060		816		153		18		1					
19		1		19		171		969		3876		11628		27132		50388		75582		92378		92378		75582		50388		27132		11628		3876		969		171		19		1			
20		1		20		190		1140		4845		15504		38760		77520		125970		167960		184756		167960		125970		77520		38760		15504		4845		1140		190		20		1	

Pascal's Triangle (rows 0–20; row number on the left)

0																						1																					
1																					1		1																				
2																				1		2		1																			
3																			1		3		3		1																		
4																		1		4		6		4		1																	
5																	1		5		10		10		5		1																
6																1		6		15		20		15		6		1															
7															1		7		21		35		35		21		7		1														
8														1		8		28		56		70		56		28		8		1													
9													1		9		36		84		126		126		84		36		9		1												
10												1		10		45		120		210		252		210		120		45		10		1											
11											1		11		55		165		330		462		462		330		165		55		11		1										
12										1		12		66		220		495		792		924		792		495		220		66		12		1									
13									1		13		78		286		715		1287		1716		1716		1287		715		286		78		13		1								
14								1		14		91		364		1001		2002		3003		3432		3003		2002		1001		364		91		14		1							
15							1		15		105		455		1365		3003		5005		6435		6435		5005		3003		1365		455		105		15		1						
16						1		16		120		560		1820		4368		8008		11440		12870		11440		8008		4368		1820		560		120		16		1					
17					1		17		136		680		2380		6188		12376		19448		24310		24310		19448		12376		6188		2380		680		136		17		1				
18				1		18		153		816		3060		8568		18564		31824		43758		48620		43758		31824		18564		8568		3060		816		153		18		1			
19			1		19		171		969		3876		11628		27132		50388		75582		92378		92378		75582		50388		27132		11628		3876		969		171		19		1		
20		1		20		190		1140		4845		15504		38760		77520		125970		167960		184756		167960		125970		77520		38760		15504		4845		1140		190		20		1	

Quick Check Answers

1. $\binom{7}{2} = \frac{7!}{2!5!} = \frac{7 \cdot 6}{2} = 21$. $\binom{7}{5} = \frac{7!}{5!2!} = \frac{7 \cdot 6}{2} = 21$. Equal!
2. Choosing 2 items from 7 determines which 5 items are left. So each selection of 2 corresponds uniquely to a selection of 5.
3. $2^{10} = 1024$
4. $2^8 = 256$ subsets
5. $1 - 5 + 10 - 10 + 5 - 1 = (1 + 10 + 5) - (5 + 10 + 1) = 16 - 16 = 0$ ✓
6. Half of $2^6 = 64$, so 32 even-sized subsets.
7. $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = 1 + 4 + 10 + 20 = 35 = \binom{7}{4}$ ✓