

MATH 8

THE BINOMIAL FORMULA — SHORT 2026–27

So far, the binomial coefficients $\binom{n}{k}$ have counted committees, subsets, grid paths, and binary strings, and they filled Pascal's triangle. Now they show up somewhere completely different: in *algebra*, when we multiply out a power like $(a + b)^n$. This is no coincidence — and seeing *why* is the heart of this handout.

A Familiar Pattern

Let's expand the first few powers of $(a + b)$ the slow way and look at the coefficients:

$(a + b)^0 = 1$	1
$(a + b)^1 = a + b$	1 1
$(a + b)^2 = a^2 + 2ab + b^2$	1 2 1
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1

The coefficients are exactly the rows of Pascal's triangle! In each term the powers of a and b add up to n , and the power of a goes down as the power of b goes up.

The Binomial Theorem

For every whole number $n \geq 0$,

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$$

The *general term* is $\binom{n}{k}a^{n-k}b^k$, for $k = 0, 1, \dots, n$.

Why It Works: a Counting Argument

$(a + b)^n$ is a product of n factors: $(a + b)(a + b) \cdots (a + b)$. To multiply it out, we pick *either* a or b from *every factor* and multiply our picks — in every possible way — and add up all the results.

A term $a^{n-k}b^k$ appears whenever we pick b from exactly k of the factors (and a from the rest). How many ways are there to choose *which* k factors give a b ? Exactly $\binom{n}{k}$. So the term $a^{n-k}b^k$ appears $\binom{n}{k}$ times, and that is its coefficient.

Example. In $(a + b)^3 = (a + b)(a + b)(a + b)$, the term a^2b comes from picking b from factor 1, or factor 2, or factor 3: baa , aba , aab . That's $\binom{3}{1} = 3$ ways, so the coefficient is 3.

Using the Formula

Watch the signs and the numbers. Treat each part of the binomial as a whole, *including its coefficient and sign*.

Example. Expand $(2x + 3)^3$, with $a = 2x$, $b = 3$:

$$\begin{aligned} (2x)^3 + 3(2x)^2(3) + 3(2x)(3)^2 + 3^3 \\ = 8x^3 + 36x^2 + 54x + 27. \end{aligned}$$

Example. Expand $(x - y)^4$, with $b = -y$: odd powers of $-y$ are negative, so the signs alternate:

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$$

Finding one term without expanding everything. Write the *general term* and choose k to get the power you want.

Example. Find the coefficient of x^5 in $(2x - 1)^8$. The general term is $\binom{8}{k}(2x)^{8-k}(-1)^k$. For x^5 we need $8 - k = 5$, so $k = 3$:

$$\binom{8}{3} \cdot 2^5 \cdot (-1)^3 = 56 \cdot 32 \cdot (-1) = -1792.$$

Example. Find the constant term of $(x^2 + \frac{1}{x})^6$. The general term is $\binom{6}{k}(x^2)^{6-k}(\frac{1}{x})^k = \binom{6}{k}x^{12-3k}$. It has no x when $12 - 3k = 0$, i.e. $k = 4$: the constant term is $\binom{6}{4} = 15$.

Making Things Cancel

Expand $(1 + \sqrt{2})^4$ and $(1 - \sqrt{2})^4$ side by side. The terms with *even* powers of $\sqrt{2}$ are the same in both (and are whole numbers); the terms with *odd* powers have opposite signs:

$$(1 + \sqrt{2})^4 = 17 + 12\sqrt{2}, \quad (1 - \sqrt{2})^4 = 17 - 12\sqrt{2}.$$

So the **sum** is 34: the square roots cancel completely! In a **difference** the opposite happens — the even-power terms cancel and the odd ones double. For example, $(x + y)^5 - (x - y)^5 = 10x^4y + 20x^2y^3 + 2y^5$.

Counting Meets Probability

Toss a fair coin 6 times. How likely are exactly 2 heads? There are $\binom{6}{2} = 15$ sequences with exactly 2 heads (choose *which* tosses are heads), and each particular sequence has probability $(\frac{1}{2})^6 = \frac{1}{64}$. So

$$P(\text{exactly 2 heads}) = \binom{6}{2} \cdot \frac{1}{64} = \frac{15}{64}.$$

The same idea works for an unfair coin: a sequence with k heads has probability $q^k p^{n-k}$, and there are $\binom{n}{k}$ of them.

Common Mistakes

- **Forgetting to raise the coefficient.** In $(2x + 3)^4$ the x^3 -term is $\binom{4}{1}(2x)^3 \cdot 3 = 96x^3$, not $4 \cdot 2x^3 \cdot 3$.
- **Losing the minus signs.** In $(a - b)^n$, the power $(-b)^k$ is negative when k is odd.
- **Mixing up k .** In $\binom{n}{k}a^{n-k}b^k$, the bottom number k matches the power of b .

Homework

1. Use the binomial formula to expand:

(a) $(x + 2y)^5$

(b) $(3a - b)^4$

(c) $(2x + 3)^5$

2. Find the coefficient of x^7 in the expansion of $(3x - 2)^{11}$.

3. Find the term that does not contain x (the *constant term*) in the expansion of $\left(x + \frac{2}{x}\right)^8$.

4. Find the coefficient of x^3 in the product $(1 + 2x)^5(1 - x)^4$. (Hint: an x^3 can be built as $x^i \cdot x^{3-i}$, one factor from each expansion.)

5. Compute $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.

6. Compute $(x + 2y)^6 - (x - 2y)^6$.

7. **M** Find the coefficient of x^4 in $(1 + x)^5(1 - x)^5$. (Hint: there is a much faster way than multiplying two expansions.)

8. **Quick estimate.** Use the first three terms of the binomial expansion of $(1 + 0.02)^{10}$ to estimate 1.02^{10} . Check with a calculator: how close did you get, and why are the remaining terms so small?

9. **M** Use the binomial theorem:

(a) Show that $7^n - 1$ is divisible by 6 for every positive integer n . (Hint: $7 = 1 + 6$.)

(b) Find the last four digits of 101^{50} .

10. **Bernoulli's inequality.** Prove that for every integer $n \geq 0$ and every real $x \geq 0$,

$$(1 + x)^n \geq 1 + nx.$$

(Hint: expand $(1 + x)^n$ using the binomial formula and notice that all the omitted terms are ≥ 0 .) When is equality achieved?

11. **H** **Which is bigger, 2024^{2025} or 2025^{2024} ?** Use the binomial theorem to show that $\left(1 + \frac{1}{n}\right)^n < 3$ for every positive integer n , and use that bound to compare the two numbers.

(Hint: divide both numbers by 2024^{2024} , then compare 2024 with $\left(1 + \frac{1}{2024}\right)^{2024}$.)

12. **Unfair coin.** You have a coin that lands tails with probability $p = \frac{1}{3}$ and heads with probability $q = \frac{2}{3}$. You toss it 8 times. Find the probability that:

(a) All 8 tosses are tails.

(b) Exactly 3 tosses are heads.

(c) At least 3 tosses are heads.

(Hint: the probability of any specific sequence with k heads and $8 - k$ tails is $q^k p^{8-k}$. Multiply by the number of such sequences.)