

MATH 8

THE BINOMIAL FORMULA — FULL 2026–27

Introduction: Expanding Powers of Binomials

So far, the binomial coefficients $\binom{n}{k}$ have appeared as the answers to counting questions — “how many ways to choose k items from n ?” — and as the entries of Pascal’s triangle. In this handout we meet them in a third, very different setting: as the coefficients that appear when you expand a power of a binomial like $(a + b)^n$.

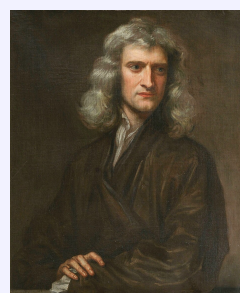
The same numbers answer both “how many ways to choose?” and “what does $(a + b)^n$ expand to?” The bridge between the two questions is the **binomial theorem**.

Historical Note

The expansion $(a + b)^n$ for whole-number n was already known to medieval algebraists — al-Karaji of Baghdad described it around 1000 CE, and the coefficients had been collected in Pascal’s triangle long before Pascal. Isaac Newton’s breakthrough, around 1665, was to ask: what if n is *not* a whole number? He discovered that, for $|x| < 1$, $(1 + x)^n$ can still be written as a (now infinite) sum

$$(1 + x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots,$$

where the formula $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ makes sense for any real n . This *generalized binomial theorem* let him write down $\sqrt{1+x}$, $\frac{1}{1+x}$, and many other functions as power series — a key step toward his invention of calculus. Newton described it in letters in 1676; the first published account appeared in a book by John Wallis in 1685.



Isaac Newton (1643–1727)

To get a feel for what’s coming, let’s first do the boring thing: expand some powers of $(a + b)$ by hand and stare at the result.

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Do you notice something? Look at the coefficients:

n	Coefficients				
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

These are exactly the rows of Pascal’s Triangle! This is no coincidence, and soon you will see *why* the same number $\binom{n}{k}$ has to appear in both places.

The Binomial Theorem

Here is the formula we noticed, stated for every power n .

The Binomial Theorem

For any non-negative integer n :

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

The *general term* is $\binom{n}{k}a^{n-k}b^k$, for $k = 0, 1, \dots, n$.

Understanding the Formula

Before we prove the formula, let's understand what it says:

- The expansion of $(a + b)^n$ has $n + 1$ terms.
- Each term has the form $\binom{n}{k}a^{n-k}b^k$ for some k from 0 to n .
- In each term, the exponents of a and b add up to n .
- The coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$ —the entry in row n , position k of Pascal's Triangle.

Example 1. Using the binomial formula, expand $(a + b)^5$.

Solution: Row 5 of Pascal's Triangle is: 1, 5, 10, 10, 5, 1.

Therefore:

$$\begin{aligned}(a + b)^5 &= 1 \cdot a^5 + 5 \cdot a^4b + 10 \cdot a^3b^2 + 10 \cdot a^2b^3 + 5 \cdot ab^4 + 1 \cdot b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Quick Check

1. Write out row 6 of Pascal's Triangle.
2. Use it to expand $(a + b)^6$.

Proof of the Binomial Theorem

Why should the coefficients be binomial coefficients? The best answer computes nothing at all: we just watch carefully what happens when the product is multiplied out, and ask “what does each coefficient *count*?”

The Combinatorial Proof

Proof. When we expand $(a + b)^n$, we are multiplying n copies of $(a + b)$:

$$(a + b)^n = \underbrace{(a + b)(a + b)(a + b) \cdots (a + b)}_{n \text{ factors}}.$$

To get a term in the expansion, we choose either a or b from each factor and multiply our choices together.

Key observation. If we choose b from exactly k of the factors (and a from the remaining $n - k$), we get the term $a^{n-k}b^k$.

How many ways can we do this? We need to choose which k of the n factors contribute a b . This can be done in $\binom{n}{k}$ ways.

Therefore the term $a^{n-k}b^k$ appears exactly $\binom{n}{k}$ times in the expansion, so its coefficient is $\binom{n}{k}$. $\square$

Example 2. Let's trace through $(a + b)^3 = (a + b)(a + b)(a + b)$.

We label the factors as 1, 2, 3. For each term, we record which factors contribute b :

Choices for b	Product	Term
none	$a \cdot a \cdot a$	a^3
factor 1	$b \cdot a \cdot a$	a^2b
factor 2	$a \cdot b \cdot a$	a^2b
factor 3	$a \cdot a \cdot b$	a^2b
factors 1, 2	$b \cdot b \cdot a$	ab^2
factors 1, 3	$b \cdot a \cdot b$	ab^2
factors 2, 3	$a \cdot b \cdot b$	ab^2
factors 1, 2, 3	$b \cdot b \cdot b$	b^3

Collecting like terms:

- a^3 appears $\binom{3}{0} = 1$ time
- a^2b appears $\binom{3}{1} = 3$ times
- ab^2 appears $\binom{3}{2} = 3$ times
- b^3 appears $\binom{3}{3} = 1$ time

So $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. ✓

The big idea

Expanding $(a + b)^n$ is secretly a counting question: the coefficient of $a^{n-k}b^k$ is the number of ways to pick which k of the n factors give a b . Whenever you see a binomial coefficient in algebra, look for the choice it is counting.

Another View: Building Each Row from the One Before

There is also a purely algebraic way to see the pattern, and it explains Pascal's rule. To get $(a + b)^3$, multiply $(a + b)^2 = a^2 + 2ab + b^2$ by a and by b , and add, lining up like terms:

$$\begin{array}{rcl}
 a \cdot (a + b)^2 & = & a^3 + 2a^2b + ab^2 \\
 b \cdot (a + b)^2 & = & a^2b + 2ab^2 + b^3 \\
 \hline
 (a + b)^3 & = & a^3 + 3a^2b + 3ab^2 + b^3
 \end{array}$$

Each new coefficient is the sum of two neighboring coefficients of the previous row: $3 = 2 + 1$, $3 = 1 + 2$. That is exactly how each row of Pascal's triangle is built from the one above it! The same thing happens at every step, from $(a + b)^n$ to $(a + b)^{n+1}$.

Quick Check

3. In the combinatorial proof, why does choosing b from exactly 2 factors out of 5 give us the term a^3b^2 ?
4. How many terms are in the expansion of $(a + b)^{10}$?

Applications of the Binomial Formula

The binomial theorem is also a workhorse. Here are five kinds of problems where it saves a lot of work: expanding messy expressions, finding one coefficient in a huge expansion, adding or subtracting “conjugate” powers so that half of the terms cancel, and computing probabilities.

Application 1: Expanding Expressions

Example 3. Expand $(x - y)^4$.

Solution: We write $(x - y)^4 = (x + (-y))^4$ and apply the binomial formula:

$$\begin{aligned}(x - y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3(-y) + \binom{4}{2}x^2(-y)^2 + \binom{4}{3}x(-y)^3 + \binom{4}{4}(-y)^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

Notice the alternating signs: when we replace b with $-y$, the terms with odd powers of y become negative.

Example 4. Expand $(2x + 3)^4$.

Solution: Let $a = 2x$ and $b = 3$:

$$\begin{aligned}(2x + 3)^4 &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(3) + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + \binom{4}{4}(3)^4 \\ &= 1 \cdot 16x^4 + 4 \cdot 8x^3 \cdot 3 + 6 \cdot 4x^2 \cdot 9 + 4 \cdot 2x \cdot 27 + 1 \cdot 81 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Application 2: Finding Specific Coefficients

Sometimes we only need one term, not the entire expansion. For example, expanding $(2x + 3)^{14}$ by hand would give us 15 separate terms — but if all we want is the coefficient of x^8 , we can read it off the binomial formula directly without computing any of the other 14.

Example 5. Find the coefficient of x^8 in the expansion of $(2x + 3)^{14}$.

Solution: The general term in the expansion is:

$$\binom{14}{k}(2x)^{14-k}(3)^k = \binom{14}{k} \cdot 2^{14-k} \cdot 3^k \cdot x^{14-k}$$

We need x^8 , so $14 - k = 8$, which gives $k = 6$.

The coefficient is:

$$\binom{14}{6} \cdot 2^8 \cdot 3^6 = 3003 \cdot 256 \cdot 729 = 560,431,872$$

Example 6. Find the coefficient of x^3y^5 in the expansion of $(x + y)^8$.

Solution: The term with x^3y^5 has the form $\binom{8}{k}x^{8-k}y^k$ where $k = 5$.

The coefficient is $\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$.

Quick Check

5. Find the coefficient of x^4 in $(x + 2)^7$.
6. Find the coefficient of a^2b^4 in $(a - b)^6$.

Application 3: Conjugate Sums and Cancellation

The next application is the kind of arithmetic trick that looks slightly miraculous at first. When two “conjugate” expressions like $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are raised to a power and added, the irrational pieces cancel *exactly*. The binomial theorem makes the cancellation completely transparent.

Example 7. Compute $(1 + \sqrt{2})^4 + (1 - \sqrt{2})^4$.

Solution: Let's expand both using the binomial formula:

$$(1 + \sqrt{2})^4 = 1 + 4\sqrt{2} + 6 \cdot 2 + 4 \cdot 2\sqrt{2} + 4 = 1 + 4\sqrt{2} + 12 + 8\sqrt{2} + 4 = 17 + 12\sqrt{2}$$

$$(1 - \sqrt{2})^4 = 1 - 4\sqrt{2} + 6 \cdot 2 - 4 \cdot 2\sqrt{2} + 4 = 1 - 4\sqrt{2} + 12 - 8\sqrt{2} + 4 = 17 - 12\sqrt{2}$$

$$\text{Adding: } (1 + \sqrt{2})^4 + (1 - \sqrt{2})^4 = (17 + 12\sqrt{2}) + (17 - 12\sqrt{2}) = 34$$

Notice how the irrational parts cancel out!

Example 8. Show that $(1 + \sqrt{3})^6 + (1 - \sqrt{3})^6$ is an integer, and find its value.

Solution: When we expand $(1 + \sqrt{3})^n$ and $(1 - \sqrt{3})^n$, terms with odd powers of $\sqrt{3}$ have opposite signs (because $(-\sqrt{3})^{\text{odd}} = -(\sqrt{3})^{\text{odd}}$), while terms with even powers have the same sign.

When we add, the odd-powered terms cancel. The remaining even powers of $\sqrt{3}$ are integers.

Let's compute:

$$\begin{aligned}(1 \pm \sqrt{3})^6 &= \binom{6}{0} + \binom{6}{1}(\pm\sqrt{3}) + \binom{6}{2}(\sqrt{3})^2 + \binom{6}{3}(\pm\sqrt{3})^3 \\ &\quad + \binom{6}{4}(\sqrt{3})^4 + \binom{6}{5}(\pm\sqrt{3})^5 + \binom{6}{6}(\sqrt{3})^6\end{aligned}$$

The even-powered terms (which survive when adding) are:

$$\begin{aligned}\binom{6}{0} \cdot 1 + \binom{6}{2} \cdot 3 + \binom{6}{4} \cdot 9 + \binom{6}{6} \cdot 27 \\ = 1 + 15 \cdot 3 + 15 \cdot 9 + 1 \cdot 27 = 1 + 45 + 135 + 27 = 208\end{aligned}$$

Since each expansion contributes these terms once, the sum is $2 \times 208 = 416$.

Therefore $(1 + \sqrt{3})^6 + (1 - \sqrt{3})^6 = 416$.

Application 4: Finding Differences

Example 9. Compute $(x + 2y)^4 - (x - 2y)^4$.

Solution: By similar reasoning, the even-powered terms in y cancel (they're the same in both expansions), while the odd-powered terms double.

The odd-powered terms in $(x + 2y)^4$ are:

$$\binom{4}{1}x^3(2y) + \binom{4}{3}x(2y)^3 = 4x^3 \cdot 2y + 4x \cdot 8y^3 = 8x^3y + 32xy^3$$

So $(x + 2y)^4 - (x - 2y)^4 = 2(8x^3y + 32xy^3) = 16x^3y + 64xy^3$.

Quick Check

7. Compute $(1 + \sqrt{2})^2 + (1 - \sqrt{2})^2$ by expanding both squares. Which terms cancel?
8. Which powers of y survive in $(x + y)^7 - (x - y)^7$? Write out the result.

Application 5: Counting Meets Probability

If a coin is tossed n times, then $\binom{n}{k}$ counts the sequences with exactly k heads (choose *which* tosses are heads). Counting tells us *how many* sequences we want; probability tells us *how much weight* each one has.

Example 10. A coin is tossed 5 times. Find the probability of exactly 3 heads if (a) the coin is fair; (b) the coin lands heads with probability $\frac{2}{3}$.

Solution: There are $\binom{5}{3} = 10$ sequences with exactly 3 heads, such as HHTHT.

(a) Each sequence of 5 tosses has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so the answer is $10 \cdot \frac{1}{32} = \frac{5}{16}$.

(b) Now a sequence with 3 heads and 2 tails, in any order, has probability $\left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{8}{243}$. All 10 such sequences have this same weight, so the answer is $10 \cdot \frac{8}{243} = \frac{80}{243}$.

Sigma Notation: Writing Sums Compactly

Many formulas in this handout are long sums written with “ $+\dots+$ ”. Mathematicians have a compact notation for such sums, which you will see in almost every advanced book. The symbol $\sum$ (the capital Greek letter *sigma*, for “sum”) means “add up”:

$$\sum_{k=0}^n a_k = a_0 + a_1 + a_2 + \dots + a_n.$$

The variable k is the **index**. It starts at the value written below the $\sum$, goes up by 1 each time until it reaches the value written above, and for each value we add the term written after the $\sum$.

Example 11. $\sum_{k=1}^5 k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$.

The index can be any letter, and it does not have to start at 0 or 1. Here are formulas we already know, written both ways:

	Written out	In sigma notation
Binomial theorem	$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n}b^n$	$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$
Row sums (Handout 2)	$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$	$\sum_{k=0}^n \binom{n}{k} = 2^n$
Gauss’s sum	$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$	$\sum_{k=1}^n k = \frac{n(n+1)}{2}$

Notice that in the binomial theorem the expression after the $\sum$ is exactly the *general term* $\binom{n}{k} a^{n-k} b^k$: sigma notation says “add the general term for every k from 0 to n .”

Example 12 (Conjugates). When $(1 + \sqrt{3})^{12} + (1 - \sqrt{3})^{12}$ is expanded, the odd powers of $\sqrt{3}$ cancel and the even powers $(\sqrt{3})^{2j} = 3^j$ survive, each doubled. Written out in full:

$$(1 + \sqrt{3})^{12} + (1 - \sqrt{3})^{12} = 2 \left[\binom{12}{0} + \binom{12}{2}3 + \binom{12}{4}3^2 + \binom{12}{6}3^3 + \binom{12}{8}3^4 + \binom{12}{10}3^5 + \binom{12}{12}3^6 \right].$$

The terms follow one pattern, $\binom{12}{2j}3^j$ for $j = 0, 1, \dots, 6$, so in sigma notation:

$$(1 + \sqrt{3})^{12} + (1 - \sqrt{3})^{12} = 2 \sum_{j=0}^6 \binom{12}{2j} 3^j = 172,928.$$

Example 13 (Probability). A coin lands heads with probability $q = \frac{2}{3}$ and tails with probability $p = \frac{1}{3}$. It is tossed 8 times. “At least 3 heads” means exactly 3, 4, 5, 6, 7, or 8 heads. Written out in full:

$$P(\text{at least 3 heads}) = \binom{8}{3}q^3p^5 + \binom{8}{4}q^4p^4 + \binom{8}{5}q^5p^3 + \binom{8}{6}q^6p^2 + \binom{8}{7}q^7p + \binom{8}{8}q^8.$$

Each term is $\binom{8}{k}q^k p^{8-k}$, for $k = 3, 4, \dots, 8$, so in sigma notation:

$$P(\text{at least 3 heads}) = \sum_{k=3}^8 \binom{8}{k} q^k p^{8-k} = \frac{2144}{2187}.$$

Quick Check

9. Write $1 + 3 + 5 + \dots + 99$ in sigma notation. (What is the k -th odd number?)
10. Write out $\sum_{k=0}^3 \binom{3}{k} x^k$. Which power of a binomial is it?

Key Takeaways

- The **binomial theorem**: $(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n}b^n$, with general term $\binom{n}{k}a^{n-k}b^k$.
- The coefficient $\binom{n}{k}$ counts the ways to choose which k of the n factors contribute a b .
- For $(a - b)^n$, terms with odd powers of b are negative. In $(a + b)^n + (a - b)^n$ the odd-powered terms cancel; in the difference the even-powered terms cancel.
- In repeated independent trials, a binomial coefficient counts how many outcome sequences have a specified number of successes.

Common Mistakes

- **Forgetting to apply exponents to coefficients**: In $(2x + 3)^4$, the term with x^3 is $\binom{4}{1}(2x)^3(3) = 96x^3$, not $4 \cdot 2x^3 \cdot 3$.
- **Wrong signs for negative terms**: In $(a - b)^n$, remember that $(-b)^k$ is negative when k is odd.
- **Off-by-one errors**: In $\binom{n}{k}a^{n-k}b^k$, the bottom number k matches the exponent of b .

Practice Problems

- Use the binomial formula to expand:
 - $(x - y)^3$
 - $(a + 3b)^3$
 - $(2x + y)^4$
 - $(x - 2)^5$
- Expand $\left(x + \frac{1}{x}\right)^4$. Which term does not contain x ?
- Expand $(1 - 2x)^4$.
- Find the coefficient of x^{10} in the expansion of $(3x + 2)^{15}$.
- Find the term containing x^4y^3 in the expansion of $(2x - y)^7$.
- Find the 4th term of the expansion of $(2a - b)^9$ (in order of decreasing powers of a).
- Find the middle term of the expansion of $(2x - 1)^6$.
- In the expansion of $(1 + x)^n$, the coefficient of x^2 is 28. Find n .
- Find the constant term in the expansion of $\left(x^2 + \frac{1}{x}\right)^6$.
- Which term of $(a + b)^{20}$ has the largest coefficient? What is that coefficient?
- Find the sum of all the coefficients in the expansion of:
 - $(3x - 2)^5$
 - $(x + 2y)^7$

(Hint: what happens when every variable equals 1?)
- Use the binomial theorem to compute 99^3 without a calculator. (Hint: $99 = 100 - 1$.)
- Write $(1 + \sqrt{2})^3$ in the form $a + b\sqrt{2}$ with integers a and b .
- Compute $(1 + \sqrt{2})^6 + (1 - \sqrt{2})^6$.
- Compute $(\sqrt{3} + \sqrt{2})^4 + (\sqrt{3} - \sqrt{2})^4$.
- Show that $(2 + \sqrt{5})^3 + (2 - \sqrt{5})^3$ is an integer, and find it.
- Is $(1 + \sqrt{2})^5 - (1 - \sqrt{2})^5$ an integer? Compute it.
- Compute $(x + 3y)^5 - (x - 3y)^5$.
- A fair coin is tossed 6 times. Find the probability of exactly 2 heads. Explain separately:
 - how many sequences have exactly 2 heads;
 - the probability of each such sequence.
- A fair die is rolled 4 times. What is the probability of getting exactly two sixes?

Practice Problems: Solutions

- $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
 - $(a + 3b)^3 = a^3 + 3a^2(3b) + 3a(3b)^2 + (3b)^3 = a^3 + 9a^2b + 27ab^2 + 27b^3$
 - $(2x + y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$
 - $(x - 2)^5 = x^5 - 5x^4(2) + 10x^3(4) - 10x^2(8) + 5x(16) - 32 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
- $(x + \frac{1}{x})^4 = x^4 + 4x^3 \cdot \frac{1}{x} + 6x^2 \cdot \frac{1}{x^2} + 4x \cdot \frac{1}{x^3} + \frac{1}{x^4} = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$. The term without x is the middle one, **6**.
- $(1 - 2x)^4 = 1 - 4(2x) + 6(2x)^2 - 4(2x)^3 + (2x)^4 = 1 - 8x + 24x^2 - 32x^3 + 16x^4$.
- The general term is $\binom{15}{k}(3x)^{15-k}(2)^k$. For x^{10} , we need $15 - k = 10$, so $k = 5$. Coefficient: $\binom{15}{5} \cdot 3^{10} \cdot 2^5 = 3003 \cdot 59049 \cdot 32 = 5,674,372,704$
- For x^4y^3 in $(2x - y)^7$: we need $(2x)^4(-y)^3$. Term: $\binom{7}{3}(2x)^4(-y)^3 = 35 \cdot 16x^4 \cdot (-y^3) = -560x^4y^3$
- The terms are $\binom{9}{k}(2a)^{9-k}(-b)^k$ for $k = 0, 1, 2, \dots$, so the 4th term has $k = 3$: $\binom{9}{3}(2a)^6(-b)^3 = 84 \cdot 64 \cdot (-1)a^6b^3 = -5376a^6b^3$.
- There are 7 terms, so the middle one is the 4th, with $k = 3$: $\binom{6}{3}(2x)^3(-1)^3 = 20 \cdot 8x^3 \cdot (-1) = -160x^3$.
- The coefficient of x^2 is $\binom{n}{2} = \frac{n(n-1)}{2} = 28$, so $n(n-1) = 56 = 8 \cdot 7$, and $n = 8$.
- The general term is $\binom{6}{k}(x^2)^{6-k}(\frac{1}{x})^k = \binom{6}{k}x^{12-3k}$. It is constant when $k = 4$: the constant term is $\binom{6}{4} = 15$.
- The coefficients are Row 20 of Pascal's triangle, which rise to the middle and then fall symmetrically. The largest is the middle one, $\binom{20}{10} = 184,756$, on the term $a^{10}b^{10}$.
- Setting every variable equal to 1 turns each term into just its coefficient, so the sum of the coefficients is the value at 1.
 - $(3 \cdot 1 - 2)^5 = 1$
 - $(1 + 2)^7 = 3^7 = 2187$
- $99^3 = (100 - 1)^3 = 100^3 - 3 \cdot 100^2 + 3 \cdot 100 - 1 = 1,000,000 - 30,000 + 300 - 1 = 970,299$.
- $(1 + \sqrt{2})^3 = 1 + 3\sqrt{2} + 3(\sqrt{2})^2 + (\sqrt{2})^3 = 1 + 3\sqrt{2} + 6 + 2\sqrt{2} = 7 + 5\sqrt{2}$.
- The even-powered terms in the expansion of $(1 + \sqrt{2})^6$ are: $\binom{6}{0} + \binom{6}{2}(\sqrt{2})^2 + \binom{6}{4}(\sqrt{2})^4 + \binom{6}{6}(\sqrt{2})^6 = 1 + 15(2) + 15(4) + 1(8) = 1 + 30 + 60 + 8 = 99$
So $(1 + \sqrt{2})^6 + (1 - \sqrt{2})^6 = 2 \times 99 = 198$
- The odd powers of $\sqrt{2}$ cancel and the even ones double: $2[(\sqrt{3})^4 + \binom{4}{2}(\sqrt{3})^2(\sqrt{2})^2 + (\sqrt{2})^4] = 2[9 + 6 \cdot 3 \cdot 2 + 4] = 2 \cdot 49 = 98$.
- Only the even powers of $\sqrt{5}$ survive, doubled: $2[2^3 + \binom{3}{2} \cdot 2 \cdot (\sqrt{5})^2] = 2[8 + 30] = 76$.
- In a *difference*, the even powers of $\sqrt{2}$ cancel and the odd ones double:

$$2\left[\binom{5}{1}\sqrt{2} + \binom{5}{3}(\sqrt{2})^3 + \binom{5}{5}(\sqrt{2})^5\right] = 2[5\sqrt{2} + 20\sqrt{2} + 4\sqrt{2}] = 58\sqrt{2}.$$

So it is *not* an integer — it is an integer multiple of $\sqrt{2}$.
- The odd-powered terms in $(x + 3y)^5$ are: $\binom{5}{1}x^4(3y) + \binom{5}{3}x^2(3y)^3 + \binom{5}{5}(3y)^5 = 5x^4(3y) + 10x^2(27y^3) + 243y^5 = 15x^4y + 270x^2y^3 + 243y^5$
So $(x + 3y)^5 - (x - 3y)^5 = 2(15x^4y + 270x^2y^3 + 243y^5) = 30x^4y + 540x^2y^3 + 486y^5$

19. (a) Choose which 2 of the 6 tosses are heads: $\binom{6}{2} = 15$ sequences.
 (b) Every particular sequence of 6 fair tosses has probability $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$. Therefore

$$P(\text{exactly 2 heads}) = \binom{6}{2} \frac{1}{2^6} = \frac{15}{64}.$$

20. Choose which 2 of the 4 rolls are sixes: $\binom{4}{2} = 6$ ways. Each such sequence has probability $\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{25}{1296}$. Total: $6 \cdot \frac{25}{1296} = \frac{25}{216} \approx \mathbf{0.116}$.

Homework

- Use the binomial formula to expand:
 - $(x + 2y)^5$
 - $(3a - b)^4$
 - $(2x + 3)^5$
- Find the coefficient of x^7 in the expansion of $(3x - 2)^{11}$.
- Find the term that does not contain x (the *constant term*) in the expansion of $\left(x + \frac{2}{x}\right)^8$.
- Find the coefficient of x^3 in the product $(1 + 2x)^5(1 - x)^4$. (*Hint: an x^3 can be built as $x^i \cdot x^{3-i}$, one factor from each expansion.*)
- Compute $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.
- Compute $(x + 2y)^6 - (x - 2y)^6$.
- M** Find the coefficient of x^4 in $(1 + x)^5(1 - x)^5$. (*Hint: there is a much faster way than multiplying two expansions.*)
- Quick estimate.** Use the first three terms of the binomial expansion of $(1 + 0.02)^{10}$ to estimate 1.02^{10} . Check with a calculator: how close did you get, and why are the remaining terms so small?
- M** Use the binomial theorem:
 - Show that $7^n - 1$ is divisible by 6 for every positive integer n . (*Hint: $7 = 1 + 6$.*)
 - Find the last four digits of 101^{50} .
- Bernoulli's inequality.** Prove that for every integer $n \geq 0$ and every real $x \geq 0$,
$$(1 + x)^n \geq 1 + nx.$$

(*Hint: expand $(1 + x)^n$ using the binomial formula and notice that all the omitted terms are ≥ 0 . When is equality achieved?*)
- H** **Which is bigger, 2024^{2025} or 2025^{2024} ?** Use the binomial theorem to show that $\left(1 + \frac{1}{n}\right)^n < 3$ for every positive integer n , and use that bound to compare the two numbers.

(*Hint: divide both numbers by 2024^{2024} , then compare 2024 with $\left(1 + \frac{1}{2024}\right)^{2024}$.*)
- Unfair coin.** You have a coin that lands tails with probability $p = \frac{1}{3}$ and heads with probability $q = \frac{2}{3}$. You toss it 8 times. Find the probability that:
 - All 8 tosses are tails.
 - Exactly 3 tosses are heads.
 - At least 3 tosses are heads.

(*Hint: the probability of any specific sequence with k heads and $8 - k$ tails is $q^k p^{8-k}$. Multiply by the number of such sequences.*)

Quick Check Answers

1. Row 6: 1, 6, 15, 20, 15, 6, 1
2. $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
3. If we choose b from exactly 2 factors out of 5, we get b^2 from those two factors and a^3 from the other three, giving a^3b^2 .
4. The expansion has $n + 1 = 10 + 1 = 11$ terms.
5. In $(x + 2)^7$, for x^4 we need $k = 3$ (so the exponent of 2 is 3). Coefficient: $\binom{7}{3} \cdot 2^3 = 35 \cdot 8 = 280$
6. For a^2b^4 in $(a - b)^6$: coefficient is $\binom{6}{4}(-1)^4 = 15 \cdot 1 = 15$
7. $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$ and $(1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$. The terms $\pm 2\sqrt{2}$ cancel, and the sum is 6.
8. Only the odd powers y, y^3, y^5, y^7 survive, each doubled: $(x + y)^7 - (x - y)^7 = 14x^6y + 70x^4y^3 + 42x^2y^5 + 2y^7$.
9. The k -th odd number is $2k - 1$, and $99 = 2 \cdot 50 - 1$, so $1 + 3 + \cdots + 99 = \sum_{k=1}^{50} (2k - 1)$. (Its value is $2500 = 50^2$.)
10. $\binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3 = 1 + 3x + 3x^2 + x^3 = (1 + x)^3$.