

MATH 8

PASCAL'S TRIANGLE AND COUNTING MODELS

Pascal's Triangle

Start with a single 1. Every new row begins and ends with 1, and each number inside is the **sum of the two numbers just above it**. That's the whole recipe — yet this little triangle, known in China, Persia, and India centuries before Blaise Pascal wrote about it in 1654, turns out to hold almost everything about counting.

Row 0					1												
Row 1					1		1										
Row 2					1		2		1								
Row 3					1		3		3		1						
Row 4					1		4		6		4		1				
Row 5					1		5		10		10		5		1		
Row 6					1		6		15		20		15		6		1

Rows and positions are counted **from 0**.
So Row 4 is 1, 4, 6, 4, 1, and its Position 2 entry is 6.

The circled entries show the rule at work:
 $4 + 6 = 10$.

A Suspicious Coincidence

In Handout 1, $\binom{n}{k}$ was the number of ways to choose k people out of n , with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Compute them for $n = 4$:

$$\binom{4}{0} = 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.$$

That's Row 4 of the triangle! And $\binom{5}{0}, \dots, \binom{5}{5}$ is Row 5. It keeps happening: **the entry in Row n , Position k is $\binom{n}{k}$** .

But why? The triangle comes from a mindless “add two neighbors” rule; binomial coefficients come from counting committees. Here's the key observation: the triangle is completely determined by its **edges** (all 1s) and its **rule** (add the two above). Binomial coefficients have the same edges — $\binom{n}{0} = \binom{n}{n} = 1$, since there's one way to choose nobody and one way to choose everybody. So we only need to show they follow the same rule.

Pascal's Identity: Why the Rule Works

Example. Count 2-person committees from Alice, Bob, Carol, Dan, and Eve by asking one question: *is Alice on it?*

Alice is on it	Alice is not on it
AB, AC, AD, AE	BC, BD, BE, CD, CE, DE
partner from the other 4: $\binom{4}{1} = 4$	both from the other 4: $\binom{4}{2} = 6$

So $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$, or $10 = 4 + 6$ — exactly the circled numbers in the triangle above!

Pascal's Identity

For $1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Why? Choose k people from n , and keep an eye on one of them, Alice. If Alice is chosen, fill the other $k - 1$ seats from the remaining $n - 1$ people: $\binom{n-1}{k-1}$ ways. If Alice is not chosen, all k seats go to the other $n - 1$ people: $\binom{n-1}{k}$ ways. Every committee is in exactly one of these groups, so we add.

The big idea

Row $n - 1$ counts committees from $n - 1$ people. Row n is what happens when **Alice joins the group**: each new committee is either an old one without her (the number above on the right) or an old one plus her (the number above on the left). Adding the two numbers above is counting “without Alice” + “with Alice.”

Same start, same edges, same rule $\implies$ the triangle and the binomial coefficients agree, row after row, forever.

Another proof, by algebra. Over the common denominator $k!(n - k)!$,

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)! \cdot k + (n-1)! \cdot (n-k)}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \binom{n}{k}.$$

It works — but the counting proof tells you *why*.

One Number, Three Costumes

The number $\binom{n}{k}$ counts three things that look completely different:

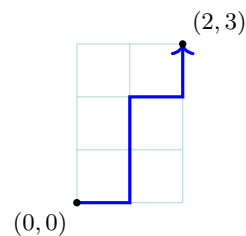
- **subsets**: ways to choose k elements from $\{1, 2, \dots, n\}$;
- **binary strings**: strings of n zeros and ones with exactly k ones;
- **grid paths**: paths of n steps, each Right (R) or Up (U), with exactly k Up steps.

They are really the *same* object in different costumes. Write a 1 in position i exactly when i is in the subset; then read the string as directions, with 0 = R and 1 = U.

Example. Take $n = 5$, $k = 3$.

$$\{2, 3, 5\} \longleftrightarrow 01101 \longleftrightarrow \text{RUURU}$$

The path ends at $(2, 3)$: 2 Rights and 3 Ups. Each costume can be translated back, so all three collections have the same size, $\binom{5}{3} = 10$.



Counting grid paths

A path from $(0, 0)$ to (a, b) uses a Rights and b Ups, $a + b$ steps in all. Choosing which b steps are Up determines the path, so there are $\binom{a+b}{b}$ paths.

Example. From $(0, 0)$ to $(4, 3)$: $\binom{7}{3} = 35$ paths.

Through a checkpoint — multiply. How many paths from $(0, 0)$ to $(5, 5)$ pass through $(2, 3)$? First get to $(2, 3)$: $\binom{5}{3} = 10$ ways. Then from $(2, 3)$ to $(5, 5)$ (3 Rights, 2 Ups): $\binom{5}{2} = 10$ ways. The two legs are chosen independently, so there are $10 \times 10 = 100$ paths.

Avoiding a point — subtract. Paths to $(5, 5)$ that *avoid* $(2, 3)$: all paths minus the ones through it, $\binom{10}{5} - 100 = 252 - 100 = 152$.

Two Patterns in the Triangle

Symmetry. Choosing the k people who go is the same as choosing the $n - k$ who stay, so

$$\binom{n}{k} = \binom{n}{n-k}.$$

That's why every row of the triangle reads the same forwards and backwards.

Row sums — a puzzle. Add up the rows: Row 2 gives $1 + 2 + 1 = 4$, Row 3 gives $1 + 3 + 3 + 1 = 8$, Row 4 gives $1 + 4 + 6 + 4 + 1 = 16$. Powers of 2! Is that a coincidence, or does it happen in every row? (One of the costumes explains it — you'll figure out which in the homework.)

Common Mistakes

- **Starting to count at 1.** Rows and positions start at 0: $\binom{4}{2} = 6$ is Row 4, Position 2.
- **Wrong numbers in the path formula.** A path to (a, b) has $a + b$ steps, so the count is $\binom{a+b}{b}$ — not $\binom{a}{b}$.
- **Adding instead of multiplying.** For a path through a checkpoint, *multiply* the two legs: every first leg can be followed by every second leg.
- **Switching the code halfway.** Decide once whether 0 means R or U, and stick with it. (Here, 0 = R and 1 = U.)

Homework

- How many paths are there from $(0, 0)$ to:
(a) $(3, 5)$? (b) $(6, 6)$? (c) $(10, 2)$?
- How many binary strings of length 10 have:
(a) exactly 4 ones?
(b) at least 8 ones?
(c) more ones than zeros?
- Prove that the sum of the entries in Row n of Pascal's triangle is 2^n . *Hint: think about binary strings.*
- Repair the triangle.** Row 9 of Pascal's triangle is 1, 9, 36, 84, 126, 126, 84, 36, 9, 1. Use only Pascal's rule to produce Row 10. Then explain why Row 10 reads the same forwards and backwards.
- Three costumes for one object.** Use $0 = R$ and $1 = U$.
(a) For the subset $S = \{2, 5, 6\}$ of $\{1, 2, \dots, 8\}$, write the matching binary string and grid path.
(b) For the binary string 10100110, write the matching subset and grid path.
- M Pascal's identity, another costume.** Prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ by counting binary strings of length n with exactly k ones. Sort them into two piles by their *last bit*. Which bit plays the role of Alice?
- Closed intersection.** A robot travels from $(0, 0)$ to $(7, 5)$ using only R- and U-moves, but the intersection $(3, 2)$ is closed for repairs. How many routes avoid it?
- The museum light board.** A display has 10 lamps, two of them red. Exactly 4 lamps must be on, but the two red lamps may not both be on. How many displays are possible?
- M** How many paths from $(0, 0)$ to $(6, 4)$ have different first and last steps? *Hint: the first and last steps are either $R \dots U$ or $U \dots R$.*
- H** A path from $(0, 0)$ to (n, n) is called **good** if it never goes above the diagonal $y = x$.
(a) List all the good paths from $(0, 0)$ to $(3, 3)$. How many are there?
(b) Can you find a formula for the number of good paths to (n, n) ? (These are the famous **Catalan numbers**.) *Hint: count the bad paths instead. A bad path first touches the line $y = x + 1$ somewhere; reflect the part of the path before that point across this line. Where does the reflected path start?*
- M No two heads in a row.** Let $f(n)$ be the number of sequences of n coin tosses in which HH never appears. For example, $f(3) = 5$: TTT, TTH, THT, HTT, HTH. (We also set $f(0) = 1$ and $f(1) = 2$.)
(a) Find $f(4)$ by listing.
(b) Find a rule that gives $f(n)$ from $f(n-1)$ and $f(n-2)$. *Hint: look at the last toss — just like we looked at Alice.*
(c) Find $f(12)$.