

# MATH 8: HANDOUT 1

## COMBINATORICS FUNDAMENTALS — SHORT 2026–27

### The Two Counting Principles

Use the **multiplication principle** for successive choices: if every first choice has  $n$  possible continuations and there are  $m$  first choices, there are  $mn$  outcomes. Use the **addition principle** for nonoverlapping cases: if one case has  $m$  outcomes and another has  $n$ , together they have  $m + n$ . Multiply within a route; add across disjoint routes.

### Factorials

For a positive integer  $n$ ,

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1, \quad 0! = 1.$$

Thus  $n!$  counts the arrangements of  $n$  distinct objects in a row. Cancel common factorial factors before multiplying large numbers. The convention  $0! = 1$  says there is exactly one way to arrange no objects: leave every position empty. It also keeps the recursion  $n! = n(n-1)!$  valid when  $n = 1$ .

### Permutations: Order Matters

The number of ways to choose and arrange  $k$  objects from  $n$  distinct objects, without repetition, is

$$P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Use permutations for ranked prizes, distinct offices, ordered codes, and other selections in which rearranging the chosen objects changes the outcome.

### Combinations: Order Does Not Matter

The number of  $k$ -element selections from  $n$  distinct objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n, k)}{k!}.$$

Use combinations for committees, hands of cards, or subsets. The division by  $k!$  removes the  $k!$  different orders of each group.

|              | Permutations                  | Combinations                         |
|--------------|-------------------------------|--------------------------------------|
| Order        | Matters                       | Does not matter                      |
| Formula      | $P(n, k) = \frac{n!}{(n-k)!}$ | $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ |
| Typical use  | Ranked roles or prizes        | Committees or subsets                |
| Relationship | $P(n, k) = \binom{n}{k} k!$   |                                      |

Each combination has  $k!$  possible orders, which explains the relationship in the table. Important edge cases are

$$P(n, 0) = 1, \quad P(n, n) = n!, \quad \binom{n}{0} = \binom{n}{n} = 1.$$

### Repeated Objects

If  $n$  objects contain groups of  $r_1, r_2, \dots, r_s$  identical objects, where  $r_1 + \cdots + r_s = n$ , then the number of distinct arrangements is

$$\frac{n!}{r_1! r_2! \cdots r_s!}.$$

The denominator removes rearrangements that look unchanged because identical copies were exchanged.

## Symmetry of Choosing

Choosing the  $k$  objects that belong to a group is equivalent to choosing the  $n - k$  objects left outside it. Hence

$$\binom{n}{k} = \binom{n}{n-k}.$$

This identity can make a calculation much shorter: choosing almost everything is the same as choosing the few objects that are omitted.

## Subsets

An  $n$ -element set has  $2^n$  subsets. For each element there are two choices: include it or exclude it. Counting the subsets by size gives

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

## Counting and Probability

When all outcomes are equally likely,

$$P(A) = \frac{\# \text{ favorable outcomes}}{\# \text{ total outcomes}}.$$

The numerator and denominator must use the same model: both ordered or both unordered. Without replacement, the number of choices decreases after each draw.

## Choosing a Counting Method

Before calculating, identify the structure of the outcome.

- If the outcome is built through successive choices, **multiply** the numbers of choices.
- If every outcome belongs to exactly one of several cases, **add** the case counts.
- If rearranging the selected objects changes the result, use a **permutation**; otherwise use a **combination**.
- If some objects are identical, divide out the orders that do not create a visible change.
- For a restriction, it may be easier to count all outcomes and subtract the forbidden ones.
- When adding cases, make sure they do not overlap; otherwise some outcomes are counted twice.

If cases overlap, redesign them so each outcome belongs to exactly one case, or correct for the overlap. This is the basic idea behind inclusion–exclusion, which appears later in the unit.

## Common Mistakes

Do not use a combination when rearranging the selection creates a different result. In  $P(n, k)$ , the product contains exactly  $k$  factors. For repeated objects, divide only by factorials of genuinely identical copies. In probability, do not count the numerator as ordered outcomes and the denominator as unordered outcomes.

## Homework

**Difficulty legend:** M marks a non-routine problem; H marks a hard or extra-credit problem. Unmarked problems may still require a good idea.

1. Compute:  
(a)  $P(8, 3)$  (b)  $\binom{9}{4}$  (c)  $\binom{12}{12}$
2. A pizza shop offers 10 toppings. How many pizzas with exactly 4 toppings are possible? What if you can order a pizza with any number of toppings (including none)?
3. How many “words” can be formed by rearranging all letters of (a) ALGEBRA and (b) STATISTICS?
4. From a group of 6 men and 4 women, a committee of 5 is to be formed. How many committees are possible? How many have exactly 3 men and 2 women? How many have at least one woman?
5. M A round table seats 8 people. In how many ways can 8 people be seated if arrangements differing only by rotation are the same? Reflections are different.
6. **Two rooks.** How many ways can two indistinguishable, nonattacking rooks be placed on an  $8 \times 8$  chessboard? (Rooks attack along a row or column. Hint: subtract the attacking placements.)
7. **Three cards.** Three cards are drawn without replacement from a standard deck. Find the probability that: (a) all are spades; (b) all are aces; (c) they are  $A\spadesuit, Q\clubsuit, K\heartsuit$  in that order; (d) those three cards appear in any order.
8. **Secret-agent codes.** A code has either two letters followed by three digits, or three letters followed by two digits. Repetition is allowed. How many codes are possible?
9. M A coach chooses a 4-person quiz team from 12 students and then chooses one team member as captain. How many outcomes are possible? Solve it in two ways.
10. H **A rotating square.** Color the four vertices of a square with three colors; colors may repeat. Colorings that differ only by rotation are the same; reflections are different. How many colorings are there?