

MATH 8: HANDOUT 1

REVIEW OF COMBINATORICS FUNDAMENTALS — 2026–27

Introduction

Welcome to Math 8! We’re starting the year with **combinatorics** — the mathematics of counting and arranging objects. “How many ways can . . . ?” is one of the oldest questions in mathematics, and one of the most surprisingly deep. A single sequence of counting numbers (like the binomial coefficients $\binom{n}{k}$) can simultaneously describe the entries of Pascal’s triangle, the coefficients of an algebraic expansion, the number of paths on a grid, and the number of subsets of a set. The same number, four different meanings — and once you start noticing this kind of coincidence, you find it everywhere.

You met the basic tools of combinatorics in Math 7: the multiplication and addition principles, factorials, permutations, and combinations. This handout is a refresher, but with a slight shift in emphasis. Instead of just rehearsing the formulas, we’ll pay attention to *when* each rule applies and *why* — the kind of careful thinking we’ll need over the next several handouts as we build toward Pascal’s triangle, the binomial theorem, and a small collection of beautiful identities that connect them. If combinatorics is new to you, don’t worry: the explanations below provide the starting point you need.

Here is a preview of where the unit is going:

- Handout 2 introduces **Pascal’s triangle** and shows that the same binomial coefficient $\binom{n}{k}$ appears in several very different-looking models.
- Handout 3 develops the **binomial theorem**, which expands $(a + b)^n$ in terms of these coefficients — the bridge between counting and algebra.
- Handout 4 explores the **properties** of Pascal’s triangle: symmetry, row sums, hockey stick, Vandermonde, divisibility patterns.
- Handout 5 collects four **advanced techniques**: pigeonhole, stars and bars, inclusion–exclusion, and derangements.
- Handout 6 reviews the whole unit through mixed problems.

You’ll meet famous mathematicians along the way — Pascal, Fermat, Dirichlet, Erdős, Ramsey — and finish the unit with a real sense of why combinatorics is one of the busiest areas of modern mathematics.

For now, let’s revisit the fundamentals.

Historical Note

Counting problems are as old as civilization. An important systematic treatment of permutations and combinations in the medieval West appears in *Maaseh Hoshav* (“The Work of Computation”), written in 1321 by the French-Jewish polymath **Levi ben Gerson**, also known as Gersonides. He described factorials, permutations, and unordered selections, and supported his formulas with careful arguments.

Another milestone came in 1654, when Blaise Pascal and Pierre de Fermat exchanged a famous series of letters about a gambling puzzle (“the Problem of Points”: how to split the stakes when a game is interrupted). Their correspondence helped launch the systematic study of **probability**, a subject that has been intertwined with combinatorics ever since.



Levi ben Gerson
(1288–1344)

The Fundamental Counting Principles

Two ideas underpin essentially every counting problem you will meet this year: the multiplication principle and the addition principle. They feel so obvious that it’s easy to use them without thinking — which is

exactly the moment to slow down and notice what's really going on. The reason these principles are so universal is that every counting problem, however complicated, can ultimately be broken into a sequence of choices (multiplication) or a list of mutually exclusive cases (addition). Mastering when to use which is the single most important skill in combinatorics.

The Multiplication Principle

If a task consists of several steps performed in sequence, multiply the number of choices at each step, provided that each partial choice has the stated number of possible continuations.

Multiplication Principle

If task 1 can be done in m ways, and for each choice, task 2 can be done in n ways, then both tasks together can be done in $m \times n$ ways.

Example 1. A restaurant offers 4 appetizers, 5 main courses, and 3 desserts. How many different three-course meals are possible?

Solution. We choose one item from each category in sequence:

$$4 \times 5 \times 3 = 60 \text{ meals.}$$

Example 2. How many 4-digit PIN codes are possible if digits can repeat?

Solution. Each of the 4 positions can be filled with any digit 0–9:

$$10 \times 10 \times 10 \times 10 = 10^4 = 10,000.$$

The Addition Principle

The multiplication principle handles “and”: do step 1 *and* step 2 *and* step 3. Its companion handles “or”: complete the task by doing option 1 *or* option 2 *or* option 3, where the options don’t overlap.

If a task can be accomplished in several *mutually exclusive* ways, add the counts.

Addition Principle

If task 1 can be done in m ways OR task 2 can be done in n ways (but not both), then the total number of ways is $m + n$.

Example 3. You can travel from A to B by 3 different flights or by 5 different train routes. How many ways can you travel?

Solution. Since you take either a flight or a train (not both):

$$3 + 5 = 8 \text{ ways.}$$

When to Multiply vs. When to Add

The danger isn’t the arithmetic — it’s deciding which structure the problem has. The words “and” and “or” can be useful clues, but they are not rules by themselves. Ask what the choices actually do.

- Use **multiplication** for a sequence of choices: after choosing at one stage, you continue to the next stage.
- Use **addition** when the possibilities have been split into *nonoverlapping cases*: every outcome belongs to exactly one case.

In the meal example above, you make three successive choices — multiplication. In the travel example, every trip is either a flight or a train, and no trip is both — addition. Some problems involve both: perhaps a trip goes through the park, with 3 ways to reach the park and 4 ways to continue, or through the museum, with 2 ways to reach it and 5 ways to continue. The total is $3 \cdot 4 + 2 \cdot 5$. First multiply within each route, then add the two disjoint cases.

Warning!

Do not add overlapping cases. If an outcome can belong to two cases, either reorganize the cases so they do not overlap or correct for the overlap. We will develop a systematic method for this in Handout 5.

Quick Check

1. A license plate has 2 letters followed by 3 digits (letters and digits may repeat). How many plates are possible?
2. You can take 4 different buses or 2 different trains to school. How many transportation options do you have?
3. A coin is flipped 6 times. How many different sequences of heads/tails are possible?

Factorials

Factorials are the natural answer to a counting question we've already implicitly seen: *how many ways are there to arrange n distinct objects in a row?* The multiplication principle tells us that the first position has n choices, the second has $n - 1$ (one object already used), the third has $n - 2$, and so on down to 1 for the last position. Multiplying these together gives a quantity important enough to deserve its own name.

The **factorial** of a positive integer n , written $n!$, is the product of all positive integers from 1 to n :

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$

The first several factorials are:

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720.$$

Factorials grow *extraordinarily* quickly. The growth rate of $n!$ is one of the great forces of combinatorics — often it makes a counting problem feel “impossible” (try arranging the 52 cards of a deck in some specific way out of all $52! \approx 8 \times 10^{67}$ orderings), and just as often it explains why a problem that looks intractable to brute force can still admit a clean closed-form answer. A few values for perspective:

$$10! = 3,628,800, \quad 15! \approx 1.3 \times 10^{12}, \quad 20! \approx 2.4 \times 10^{18}.$$

By $n = 25$ we have $25! \approx 1.5 \times 10^{25}$ — already more than the estimated number of stars in the observable universe.

Why $0! = 1$

By definition, $0! = 1$. This might seem strange, but there are good reasons:

- **Combinatorial:** There is exactly one way to arrange zero objects: do nothing!
- **Recursive:** Since $n! = n \times (n - 1)!$, setting $n = 1$ gives $1! = 1 \times 0!$, so $0! = 1$.
- **Formula consistency:** The formulas for permutations and combinations require $0! = 1$ to work correctly.

Quick Check

4. Compute: $\frac{8!}{6!}$
5. Compute: $\frac{10!}{7! \cdot 3!}$
6. Simplify: $\frac{(n+1)!}{n!}$

Permutations (Order Matters)

When we count arrangements — standing in line, sitting around a table, awarding ranked prizes — the order of the objects is part of the answer. Putting Alice in first place and Bob in second is a different arrangement from putting Bob in first and Alice in second, even though the same two people are involved. Counting problems where order matters are called **permutations**.

A **permutation** is an arrangement of distinct objects, chosen without repetition, where **order matters**.

Arranging All Objects

The number of ways to arrange n distinct objects in a line is:

$$n!$$

Example 4. In how many ways can 5 students stand in a line for a photo?

Solution. The first position can be filled by any of 5 students, the second by any of 4 remaining, and so on:

$$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120.$$

Arranging Some Objects

For integers $0 \leq k \leq n$, the number of ways to choose and arrange k objects from n distinct objects, without repetition, is:

$$P(n, k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}.$$

Example 5. From 10 runners, how many ways can we award gold, silver, and bronze medals?

Solution. We need to arrange 3 runners from 10 (order matters because medals are different):

$$P(10, 3) = 10 \times 9 \times 8 = 720.$$

Why the Formula Works

Think of filling k positions one at a time. Position 1 has n choices, position 2 has $n-1$ choices, position 3 has $n-2$ choices, and so on, until position k has $n-k+1$ choices. Multiplying gives

$$n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Quick Check

7. Compute $P(7, 3)$.
8. In how many ways can a club of 12 choose a president, vice-president, and secretary?
9. Verify: $P(n, n) = n!$

Combinations (Order Does Not Matter)

Sometimes the order of the chosen objects *doesn't* matter. A committee of three people doesn't change identity if we list the members in a different order; a set of three pizza toppings doesn't become a different pizza if we name the toppings in a different sequence. Counting problems where order doesn't matter are called **combinations**, and they are governed by the binomial coefficients — the most important sequence of numbers in this entire unit.

A **combination** is a selection of distinct objects, without repetition, where **order does not matter**.

The Binomial Coefficient

For integers $0 \leq k \leq n$, the number of ways to choose k objects from n distinct objects, without repetition and without regard to order, is called a **binomial coefficient**:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Example 6. From 7 students, how many ways can we form a committee of 3?

Solution. Order doesn't matter (a committee is just a set of people):

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{210}{6} = 35.$$

Why We Divide by $k!$

Consider choosing 3 people from 7 for a committee.

If order mattered (like assigning President, VP, Secretary), there would be:

$$P(7, 3) = 7 \times 6 \times 5 = 210 \text{ ways.}$$

But order doesn't matter. Each set of 3 people appears $3! = 6$ times among those 210 arrangements (once for each ordering). So we divide:

$$\binom{7}{3} = \frac{P(7, 3)}{3!} = \frac{210}{6} = 35.$$

Key Insight

$$\binom{n}{k} = \frac{P(n, k)}{k!} = \frac{\text{permutations}}{\text{orderings of each group}}$$

Arrangements with Repeated Objects

Sometimes we arrange objects that are not all distinct. If a word has n letters, with repetition counts $r_1, r_2, \dots, r_s$ satisfying $r_1 + r_2 + \cdots + r_s = n$, then

$$\text{number of distinct arrangements} = \frac{n!}{r_1! r_2! \cdots}.$$

Why divide? If we temporarily label the repeated copies, we get $n!$ arrangements, but relabeling the r_i identical copies of one type does not create a new word. Thus every visible arrangement was counted $r_1! r_2! \cdots$ times.

For example, BOOK has $4!$ arrangements if its two O's are temporarily distinguished, but swapping those O's changes nothing. Therefore it has $4!/2! = 12$ distinct arrangements. Handout 3 will develop this idea further as part of *multinomial coefficients*.

Reading Binomial Coefficients

The notation $\binom{n}{k}$ is read “ n choose k .” Some books write ${}_nC_k$ or $C(n, k)$.

Quick Check

10. Compute $\binom{8}{2}$.
11. Compute $\binom{10}{7}$.
12. A pizza shop offers 8 toppings. How many 3-topping pizzas are possible?

Counting all subsets at once: 2^n

So far we’ve counted subsets of a *fixed* size. What happens if we don’t care about the size — if we want to count *all* subsets of an n -element set at once, including the empty set and the whole set? There’s a quick way to do this, and it produces one of the most famous identities in elementary combinatorics.

If we add up the number of subsets of every possible size, we get the total number of subsets:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (\text{total subsets of an } n\text{-element set}).$$

There is a much faster way to count subsets directly. To specify a subset of $\{1, 2, \dots, n\}$, decide for each element whether to *include* it or *exclude* it. Each of the n elements has 2 independent choices, so by the multiplication principle the total number of subsets is

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ factors}} = 2^n.$$

Total number of subsets

A set of n elements has exactly 2^n subsets, including the empty set and the whole set. Equivalently,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

For instance, a 3-element set $\{a, b, c\}$ has $2^3 = 8$ subsets: $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. We will see this identity again in Handout 4 (Properties of Pascal’s Triangle), where it appears as the “row sum” identity.

Comparing Permutations and Combinations

Most counting problems you’ll encounter reduce, sooner or later, to choosing whether order matters — which means choosing between permutations and combinations. The two are very closely related: a combination is a permutation that has “forgotten” the order of its elements. Concretely, every k -element combination corresponds to $k!$ permutations of those same elements, which is why the binomial coefficient $\binom{n}{k}$ equals $P(n, k)$ divided by $k!$. The table below records the contrast.

	Permutations	Combinations
Order	Matters	Does not matter
Formula	$P(n, k) = \frac{n!}{(n - k)!}$	$\binom{n}{k} = \frac{n!}{k!(n - k)!}$
Example	Awarding 3 medals from 10	Choosing 3 people from 10
Relationship	$P(n, k) = \binom{n}{k} \times k!$	

Example 7. From 5 people (Alice, Bob, Carol, Dan, Eve), we select 2.

- (a) How many ways if they receive different prizes (1st and 2nd place)?
- (b) How many ways if they form a committee (no roles)?

Solution.

(a) Order matters: $P(5, 2) = 5 \times 4 = 20$.

(b) Order doesn't matter: $\binom{5}{2} = \frac{20}{2!} = 10$.

Indeed, $20 = 10 \times 2$ because each pair appears twice in the ordered count: (Alice, Bob) and (Bob, Alice).

A Counting Reminder for Probability

When a finite experiment has equally likely outcomes,

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}.$$

You may count ordered outcomes or unordered outcomes, but the numerator and denominator must use the same model. When objects are drawn *without replacement*, the number of available choices decreases after each draw.

Key Takeaways

- **Multiplication principle:** For sequential tasks, multiply the counts.
- **Addition principle:** For nonoverlapping cases, add the counts.
- **Factorial:** $n! = n(n-1) \cdots 2 \cdot 1$, with $0! = 1$.
- **Permutations** (order matters): $P(n, k) = \frac{n!}{(n-k)!}$
- **Combinations** (order doesn't matter): $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Key relationship:** $P(n, k) = \binom{n}{k} \times k!$
- **Repeated objects:** divide $n!$ by the factorial of each repetition count.
- **Equally likely outcomes:** probability is favorable outcomes divided by all outcomes.

Common Mistakes

- **Confusing when order matters:** Ask yourself: "Would rearranging the selection give something different?" If yes, use permutations; if no, use combinations.
- **Forgetting to divide by $k!$:** When order doesn't matter, each group is counted $k!$ times in the permutation count.
- **Off-by-one errors:** In $P(n, k)$, the product has exactly k factors: $n, n-1, \dots, n-k+1$.
- **Mixing up n and k :** Always identify n (total objects) and k (objects chosen) clearly.

Practice Problems

The problems below are a mix of familiar Math 7 territory and a few that push slightly further. For each one, before you reach for a formula, ask yourself two questions: (1) *am I making sequential choices (multiply) or splitting outcomes into nonoverlapping cases (add)?* and (2) *does the order of the chosen objects matter (permutation) or not (combination)?* Those questions usually determine the first move.

1. A club of 15 people needs to choose a chair, a treasurer, and a secretary. In how many ways can this be done?
2. The same club of 15 people needs to choose a 3-person committee (no specific roles). In how many ways can this be done?
3. In a meeting of 20 people, everyone shakes hands once with everyone else. How many handshakes occur?
4. How many different “words” can be formed by rearranging all the letters of MATH? Of BOOK? Of MISSISSIPPI?
5. A test has 10 questions. A student must answer exactly 7 of them.
 - (a) In how many ways can the student choose which 7 questions to answer?
 - (b) If the first 3 questions are mandatory, in how many ways can the student choose the remaining questions?
6. From a standard deck of 52 cards, how many 5-card hands contain:
 - (a) all hearts?
 - (b) exactly 3 aces?
7. A robot travels from its laboratory to a charging station. It can go through the park, with 3 ways to reach the park and 4 ways to continue, or through the museum, with 2 ways to reach the museum and 5 ways to continue. The two kinds of route never overlap. How many routes are possible?

Practice Problems: Solutions

1. **Chair, treasurer, secretary from 15.**
Order matters (different roles), so we use permutations:

$$P(15, 3) = 15 \times 14 \times 13 = 2730.$$

2. **3-person committee from 15.**
Order doesn't matter (just a set of people):

$$\binom{15}{3} = \frac{15 \times 14 \times 13}{3!} = \frac{2730}{6} = 455.$$

3. **Handshakes among 20 people.**
Each handshake involves 2 people, and order doesn't matter:

$$\binom{20}{2} = \frac{20 \times 19}{2} = 190.$$

4. **Rearrangements of letters.**

- MATH: 4 distinct letters, so $4! = 24$.
- BOOK: 4 letters with O repeated twice: $\frac{4!}{2!} = 12$.

- MISSISSIPPI: 11 letters with M(1), I(4), S(4), P(2): $\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = \frac{39916800}{1 \cdot 24 \cdot 24 \cdot 2} = 34650$.

5. Choosing 7 questions from 10.

(a) $\binom{10}{7} = \binom{10}{3} = \frac{10 \times 9 \times 8}{6} = 120.$

(b) With first 3 mandatory, student chooses 4 more from remaining 7: $\binom{7}{4} = 35$.

6. 5-card hands from 52 cards.

(a) All hearts: choose 5 from 13 hearts: $\binom{13}{5} = 1287$.

(b) Exactly 3 aces: choose 3 of 4 aces, then 2 from remaining 48 non-aces:

$$\binom{4}{3} \times \binom{48}{2} = 4 \times 1128 = 4512.$$

7. **Robot routes.** Multiply within each route and add the two nonoverlapping cases:

$$3 \cdot 4 + 2 \cdot 5 = 12 + 10 = 22.$$

Homework

Difficulty legend: M marks a non-routine problem; H marks a hard or extra-credit problem. Unmarked problems may still require a good idea.

1. Compute:
(a) $P(8, 3)$ (b) $\binom{9}{4}$ (c) $\binom{12}{12}$
2. A pizza shop offers 10 toppings. How many pizzas with exactly 4 toppings are possible? What if you can order a pizza with any number of toppings (including none)?
3. How many “words” can be formed by rearranging all letters of:
(a) ALGEBRA
(b) STATISTICS
4. From a group of 6 men and 4 women, a committee of 5 is to be formed.
(a) How many committees are possible?
(b) How many have exactly 3 men and 2 women?
(c) How many have at least one woman?
5. **M** A round table seats 8 people. In how many ways can 8 people be seated if two arrangements that differ only by rotation are considered the same? Reflected arrangements are considered different.
6. **Two rooks on a chessboard.** An 8×8 chessboard has 64 squares. Two *rooks* attack each other if they share a row or a column. How many ways are there to place two *indistinguishable* rooks on the board so that they do *not* attack each other?
(Hint: count placements where they *do* attack, and subtract.)

7. **Three cards from a deck.** Three cards are drawn at random, without replacement, from a standard 52-card deck (4 suits, 13 values). Find the probability of each event:
- (a) All three are spades.
 - (b) All three are aces.
 - (c) The cards are $A\spadesuit, Q\spadesuit, K\spadesuit$ in that order.
 - (d) The cards are $A\spadesuit, Q\spadesuit, K\spadesuit$ in any order.
8. **Secret-agent codes.** A code uses one of two formats: two letters followed by three digits, or three letters followed by two digits. Repetition is allowed. How many codes are possible?
9. **M** A coach chooses a 4-person quiz team from 12 students and then chooses one of the four team members as captain. How many outcomes are possible? Solve it in two different ways.
10. **H** **A rotating square.** Color the four vertices of a square using three colors; colors may repeat. Two colorings are considered the same if one is obtained from the other by rotating the square. Reflections are considered different. How many colorings are there?

Quick Check Answers

Counting Principles

1. $26^2 \cdot 10^3 = 676,000$ plates.
2. $4 + 2 = 6$ transportation options.
3. $2^6 = 64$ sequences.

Factorials

4. $\frac{8!}{6!} = 8 \cdot 7 = 56$.
5. $\frac{10!}{7!3!} = 120$.
6. $\frac{(n+1)!}{n!} = n + 1$.

Permutations

7. $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$.
8. $P(12, 3) = 12 \cdot 11 \cdot 10 = 1320$.
9. $P(n, n) = n!$; both count arrangements of all n objects.

Combinations

10. $\binom{8}{2} = 28$.
11. $\binom{10}{7} = \binom{10}{3} = 120$.
12. $\binom{8}{3} = 56$.