MATH 6

LOGIC III: IMPLICATION. LOGICAL LAWS — SHORT 2026–27

"If it rains, then the ground is wet." "If you finish your homework, you get ice cream." Sentences built with *if . . . then* are everywhere — in rules, in promises, and in every mathematical theorem. More than 2,300 years ago Aristotle studied how such sentences fit together into arguments, and in the 1800s Lewis Carroll (the author of *Alice in Wonderland*) turned them into delightfully silly logic puzzles. Today we learn how they really work.

Implication: A Promise

"If P, then Q" is written $P \to Q$. Think of it as a **promise**. A parent says: "If you do your homework, then you get ice cream." When is the promise *broken*?

- Homework done, ice cream given: promise **kept**.
- Homework done, *no* ice cream: promise **broken**!
- No homework, ice cream anyway (maybe it's a birthday): the promise only said what happens *if* you do your homework — not broken.
- No homework, no ice cream: not broken either.

A promise that wasn't broken counts as *true*. That gives the truth table:

P	Q	$P \to Q$
T	T	T
T	F	**F**
F	T	T
F	F	T

The one way to break a promise

$P \to Q$ is false *only* when P is true and Q is false.

Careful: the arrow has a direction. The **converse** $Q \to P$ is the promise read backwards, and it can say something completely different. "If it is a dog, then it has four legs" is true; "if it has four legs, then it is a dog" is false — cats exist! When a promise works in *both* directions we say "P **if and only if** Q" and write $P \leftrightarrow Q$.

Three Useful Laws

Two statements are **equivalent** ($\equiv$) if they are true in exactly the same situations. A truth table can prove it: just check every row.

- **Contrapositive:** $P \to Q \;\equiv\; \neg Q \to \neg P$. "If it rains, the ground is wet" means the same as "if the ground is *not* wet, it did *not* rain."
- **Implication law:** $P \to Q \;\equiv\; \neg P \vee Q$. "If I study, I pass" means "either I don't study, or I pass."
- **De Morgan's laws:** $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ and $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$. "I did *not* do both chores" means "I skipped at least one."

Watch out: the **inverse** $\neg P \to \neg Q$ ("if you don't do your homework, no ice cream") is *not* equivalent to $P \to Q$.

Chains of Reasoning

Implications can be linked like dominoes: if $P \to Q$ and $Q \to R$, then $P \to R$.

Example. Suppose: if I finish my science project, I can go to the class party; if I go to the party, I get cake; if I get cake, I will be smiling. At the end of the day, I am *not* smiling. What happened?

Walk the chain backwards using contrapositives: not smiling $\Rightarrow$ no cake $\Rightarrow$ no party $\Rightarrow$ **I did not finish my project.** (We write $\Rightarrow$ for "so we can conclude"; the single arrow $\to$ is part of a statement.)

Knights and Knaves Make Promises Too

On the Island of Knights and Knaves, implications lead to some surprising conclusions.

Example. A says: "If I am a Knight, then B is a Knight."

If A were a Knave, the "if" part ("I am a Knight") would be false — so the whole promise would be *true*, and a Knave can't say something true. So A is a **Knight**. Then the promise is true and its "if" part holds, so its "then" part holds too: **B is a Knight**.

Common Mistakes

- **Reading the arrow backwards:** $P \to Q$ does not mean $Q \to P$.
- **Forgetting the "empty promise":** when P is false, $P \to Q$ is true, whatever Q is.
- **Mixing up contrapositive and inverse:** $\neg Q \to \neg P$ is equivalent to $P \to Q$; $\neg P \to \neg Q$ is not.

Homework

1. If today is Thursday, then Jane's class goes to the library. If Jane's class goes to the library, then she brings home new library books. Today, Jane brought home no new library books. What can you conclude?

2. A teacher says: "If you do not take the final exam, you will get an F." Does this mean

   (a) If the student *does* take the final exam, they will *not* get an F?

   (b) If the student *does not* get an F, must they have taken the exam?

3. Many trucks carry the warning:

   > "If you do not see my mirrors, then I do not see you."

   Rewrite this message in an equivalent form **without** using the word "not."

4. Lewis Carroll puzzle:

   (a) All hummingbirds are richly colored.

   (b) No large birds live on honey.

   (c) Birds that do not live on honey are dull in color.

   What conclusion logically follows?

5. Babies, crocodiles, and logic (Lewis Carroll):

   (a) All babies are illogical.

   (b) Nobody is despised who can manage a crocodile.

   (c) Illogical persons are despised.

   What is the natural conclusion?

6. On the Island of Knights and Knaves, two inhabitants X and Y say:

   - X: "If Y is a knave, then I am a knave."
   - Y: "If X is a knave, then I am a knave."

   Determine who is a knight and who is a knave.

7. Carl says:

   > "If Sue is a knave, then this road leads to the capital."

   Later you learn that Carl himself is a knave. What can you conclude about the road?

8. M On the neighboring island, there are three kinds of people: **knights** (always true), **knaves** (always false), **normals** (sometimes true, sometimes false). Among A, B, and C, one is a knight, one is a knave, and one is normal. They say:

   - A: "B is normal."
   - B: "C is a knave."
   - C: "If I am a knight, then A is normal."

   Determine who is who.

9. M Back on the Island of Knights and Knaves (no Normals!), you meet three inhabitants A, B, and C. A says:

   "If I am a knight, then B and C are both knaves."

   What are A, B, and C?

10. M On the Island of Knights and Knaves, A says:

    "We are of the same type if and only if B is a Knight."

    What are A and B? Find every possible assignment.

11. Verify using a truth table that

$$A \to B \equiv \neg(A \wedge \neg B).$$

    Explain in words why this makes sense.

12. Write truth tables for the following formulas. Are they equivalent?

    (a) $(A \vee B) \wedge (A \vee C)$

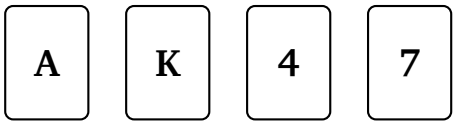
    (b) $A \vee (B \wedge C)$

13. Show that

$$(P \to Q) \vee (Q \to P)$$

    is always true. Explain in words why at least one of the two implications must hold.

14. **The Four-Card Puzzle.** Each card has a letter on one side and a number on the other. Four cards lie on the table:



    The rule is: "*If a card has a vowel on one side, then it has an even number on the other side.*" Which cards do you *have* to turn over to check whether the rule is being followed? Turn over as few as possible, and explain why each of the other cards doesn't matter.

    (When this puzzle is given to adults, most of them get it wrong!)