MATH 6

LOGIC III: IMPLICATION. LOGICAL LAWS — FULL 2026–27

Implication: "If . . . , then . . . "

Historical Note: Implications and Syllogisms

Long before modern algebra or geometry were formalized, people were already studying how to make correct arguments. The first complete system of logical reasoning comes from **Aristotle** (384–322 BC), who invented what we now call *syllogistic logic*. A syllogism is a type of argument built from statements of the form "If . . . , then . . . "—what we now call **implications**.
A classic Aristotelian syllogism looks like:

All humans are mortal.
Socrates is a human.
Therefore, Socrates is mortal.

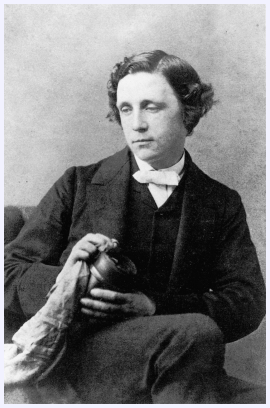
Aristotle (384–322 BC)

The conclusion follows not because of the topic (humans or mortality), but because of the *logical structure*: if **All** A are B, and C is an A, then C must be B. Aristotle spent much of his work on logic classifying such patterns and determining which ones are valid.

Implication as a logical rule. In modern notation we write statements such as "If P, then Q" as

$$P \to Q.$$

Although Aristotle treated implications through language, later logicians replaced words with symbols, allowing precise rules and truth tables. This symbolic treatment was developed much later, in the 19th century, but its roots lie in Aristotelian syllogisms.



Lewis Carroll
(1832–1898)

Lewis Carroll and logical puzzles. Jump ahead more than two thousand years to **Lewis Carroll** (Charles Dodgson), the author of *Alice in Wonderland*. Carroll was also a mathematician fascinated by logic. He invented many whimsical puzzles that use implications in unexpected ways. For example:

All babies are illogical.
Illogical persons are despised.
Nobody is despised who can manage a crocodile.
What can we conclude?

These puzzles show how implication chains can lead to surprising results when combined carefully—exactly the same ideas used in formal proofs today.

From Aristotle to modern logic. Aristotle introduced structured reasoning. Medieval scholars preserved and expanded it. Modern mathematicians such as Boole, De Morgan, and Frege turned implications into algebraic rules, and today we express them with truth tables. Every "If P, then Q" statement we use in mathematics and computer science traces back to the ancient study of syllogisms.

So far we have seen NOT, AND, and OR. Another very important logical operation is **implication**, usually read as "if . . . , then . . . ".

If P and Q are propositions, we write

$$P \to Q$$

and read it "If P, then Q." When is such a statement true? Think of it as a *promise*, and ask: in which situations is the promise broken?

A Promise: When Is It Broken?

A parent says:

> *"If you do your homework, then you get ice cream."*

Here P = "you do your homework" and Q = "you get ice cream." Let's judge all four situations.

1) **Homework done, ice cream given** (P true, Q true). The promise is **kept**.
2) **Homework done, no ice cream** (P true, Q false). You did your part and got nothing — the promise is **broken**.
3) **No homework, but ice cream anyway** (P false, Q true). Maybe it's a birthday. The promise only said what happens *if* you do your homework, so it was not broken.
4) **No homework, no ice cream** (P false, Q false). Again the condition never happened, so the promise was not broken.

A promise that was not broken counts as *true*. Writing our four verdicts in a table gives the truth table of implication:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication rule. The statement $P \to Q$ is false only when P is true and Q is false. In every other case, $P \to Q$ is true.

Quick Check

1. What is the truth value of $P \to Q$ when P is false and Q is true?
2. If "It is raining $\to$ I carry an umbrella" is true, and it IS raining, what can you conclude?
3. If "It is raining $\to$ I carry an umbrella" is true, and I am NOT carrying an umbrella, what can you conclude?
4. Is $P \to Q$ the same as $Q \to P$?

A two-way promise: "if and only if"

The statement $Q \to P$ is called the **converse** of $P \to Q$: the same promise read backwards. Sometimes a promise works in *both* directions: "If you finish your vegetables, you get dessert — *and* if you get dessert, it means you finished your vegetables." We say this in one go: "You get dessert **if and only if** you finish your vegetables." In symbols we write $P \leftrightarrow Q$, which means $(P \to Q) \wedge (Q \to P)$: both arrows at once.

Some Laws of Logic

Different logical expressions can sometimes mean the same thing in all situations. When two expressions always have the same truth value (for every possible choice of P, Q, etc.), we say they are **logically equivalent**.

We write

$$A \equiv B$$

to mean that statements A and B are logically equivalent.

Examples of Logical Laws

Here are some important logical laws:

- **Commutative laws:**

  $$P \wedge Q \equiv Q \wedge P, \qquad P \vee Q \equiv Q \vee P.$$

  Examples:

  - AND: "I cleaned my room **and** finished my homework" means the same as "I finished my homework **and** cleaned my room."
  - OR: "It is cold **or** it is windy" (maybe both!) is the same as "It is windy **or** it is cold."

- **De Morgan's laws:**

  $$\neg(P \wedge Q) \equiv \neg P \vee \neg Q, \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

  Examples:

  - $\neg(P \wedge Q)$: "I will **not** do both chores" means "I will **not** walk the dog **or** I will **not** take out the trash" (at least one chore won't happen).
  - $\neg(P \vee Q)$: "I did **not** watch TV **or** play video games" means "I **neither** watched TV **nor** played video games."

- **Implication law:**

  $$P \to Q \equiv \neg P \vee Q.$$

  Example:

  - "If I study, then I pass the test" means "Either I do **not** study, **or** I pass." The only time the implication fails is when I study but fail.

- **Contrapositive law:**

  $$P \to Q \;\equiv\; \neg Q \to \neg P.$$

  Example:

  - "If it is raining, then the ground is wet" is equivalent to "If the ground is **not** wet, then it is **not** raining."

We can *prove* these laws using truth tables.

De Morgan's Law with a Truth Table

You already met this law in Handout 4. Now let us prove it with a truth table:

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

We build a truth table with columns for both sides:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

The last two columns, $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$, have exactly the same truth values (F, T, T, T). Therefore the two statements are logically equivalent:

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

Implication as "(NOT P) OR Q"

We now prove the implication law:

$$P \to Q \equiv \neg P \vee Q.$$

We compare $P \to Q$ and $\neg P \vee Q$ in a truth table:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The column for $P \to Q$ and the column for $\neg P \vee Q$ are identical (T, F, T, T). So the two expressions are logically equivalent:

$$P \to Q \equiv \neg P \vee Q.$$

This is very useful: every "if ..., then ..." statement can be rewritten using only NOT and OR.

Contrapositive Law

We now prove the **contrapositive**:

$$P \to Q \;\equiv\; \neg Q \to \neg P.$$

The statement on the right, $\neg Q \to \neg P$, is called the *contrapositive* of $P \to Q$. We claim that these two statements are logically equivalent.
We prove this using a truth table. Compare the truth values of $P \to Q$ and $\neg Q \to \neg P$:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The last two columns are identical (T, F, T, T). Therefore,

$$P \to Q \equiv \neg Q \to \neg P.$$

This means that saying "If P, then Q" is logically the same as saying "If not Q, then not P". This is why, in proofs, we are often allowed to argue using the contrapositive instead of the original implication.

Quick Check

5. Write the contrapositive of: "If it snows, school is canceled."
6. Using De Morgan's law, simplify $\neg(P \vee Q)$.
7. Is $P \to Q$ the same as $\neg P \to \neg Q$? (This is called the "inverse.")

Using Implication to Deduce New Facts

Logical implication is especially powerful because it allows us to *deduce new results from known ones*. Here are some of the most important rules of inference:

- **Transitivity:** If $A \to B$ and $B \to C$, then we may conclude
$$A \to C.$$
- **Contrapositive inference:** If $A \to B$ is true and B is false, then A must also be false.
- **Proof by contradiction:** If assuming A leads logically to something impossible, then A cannot be true — the assumption must be wrong.

These rules let us build longer chains of reasoning from simpler pieces.

Two different arrows

The single arrow $\to$ is part of a *statement*: "$A \to B$" means "if A, then B." The double arrow $\Rightarrow$ is part of our *reasoning*: it means "so we can conclude." For example,

$$\neg\text{Smile} \Rightarrow \neg\text{Cake}$$

reads "I am not smiling, *so we can conclude* that I did not get cake."

A Chain of Reasoning in Everyday Life

Suppose the following statements are true:

1) If I finish my science project, then I can go to the class party. (Project $\to$ Party)
2) If I go to the class party, then I get a slice of cake. (Party $\to$ Cake)
3) If I get a slice of cake, then I will be smiling. (Cake $\to$ Smile)

At the end of the day, you see me and notice:

I am *not* smiling.

What can you say about me?

Using logical rules, we can work backward and deduce what must have happened.

Step 1: From "Cake $\to$ Smile" and the fact that I am not smiling, we can conclude

$$\neg\text{Smile} \Rightarrow \neg\text{Cake}.$$

(Contrapositive inference.)

Step 2: From "Party $\to$ Cake" and the fact that I did not get cake, we obtain

$$\neg\text{Cake} \Rightarrow \neg\text{Party}.$$

Step 3: From "Project $\rightarrow$ Party" and the fact that I did not go to the party,

$$\neg\text{Party} \Rightarrow \neg\text{Project}.$$

Therefore, one simple observation (that I was not smiling) allows us to conclude:

$$\boxed{\text{I did not get cake, I did not go to the party, and I did not finish my project.}}$$

Notice what we did: we walked *backward* along a chain of implications, using contrapositive inference three times. One small observation at the end of the chain told us about everything before it.

Can There Be a Pet Dragon?

Consider the following statements about creatures:

1) Every dragon hoards treasure. (Dragon $\rightarrow$ Treasure)
2) Anyone who hoards treasure frightens villagers. (Treasure $\rightarrow$ Frightens)
3) No creature that frightens villagers is harmless. (Frightens $\rightarrow \neg$Harmless)
4) All pets are harmless. (Pet $\rightarrow$ Harmless)
5) Draco is a dragon.

Can Draco the dragon be a pet?

Step 1: Follow the chain for Draco. By (5), Draco is a dragon. By (1), Draco hoards treasure. By (2), Draco frightens villagers. By (3), Draco is *not* harmless:

$$\text{Dragon} \Rightarrow \text{Treasure} \Rightarrow \text{Frightens} \Rightarrow \neg\text{Harmless}.$$

Step 2: Use the "pet" information. Statement (4) says Pet $\rightarrow$ Harmless. Its contrapositive is $\neg$Harmless $\rightarrow \neg$Pet. Draco is not harmless, so Draco *is not* a pet.

Conclusion.
There cannot exist a creature that is both:

- a dragon that hoards treasure and frightens villagers, *and*
- a pet that is harmless.

In other words, under the given assumptions, a "pet dragon" like Draco is logically impossible.

Tea Party Puzzle

At a very proper afternoon tea, the following statements are all true:

1) Every guest who likes jam for tea is invited to the garden party. (Jam $\rightarrow$ Invited)
2) Every guest who is invited to the garden party is invited again next week. (Invited $\rightarrow$ InvTwice)
3) No guest who arrives late is ever invited twice. (Late $\rightarrow \neg$InvTwice)
4) Every guest who talks loudly at tea arrives late. (Loud $\rightarrow$ Late)

What can we conclude from these statements?

Solution using implication chains.

Let

$$\text{Jam} = \text{“likes jam for tea”}, \quad \text{Invited} = \text{“is invited to the garden party”},$$

$$\text{InvTwice} = \text{“is invited again next week”}, \quad \text{Late} = \text{“arrives late”}, \quad \text{Loud} = \text{“talks loudly at tea”}.$$

From (1) and (2):

$$\text{Jam} \to \text{Invited} \quad \text{and} \quad \text{Invited} \to \text{InvTwice} \quad \Rightarrow \quad \text{Jam} \to \text{InvTwice}.$$

From (4) and (3):

$$\text{Loud} \to \text{Late} \quad \text{and} \quad \text{Late} \to \neg\text{InvTwice} \quad \Rightarrow \quad \text{Loud} \to \neg\text{InvTwice}.$$

So we have two chains:

$$\text{Jam} \to \text{InvTwice}, \qquad \text{Loud} \to \neg\text{InvTwice}.$$

We can rewrite the second statement using contrapositive as

$$\text{InvTwice} \to \neg\text{Loud}.$$

Combining it with the first statement, gives us the following implication:

$$\text{Jam} \to \neg\text{Loud}$$

that is,

No guest who likes jam for tea talks loudly at tea.

Key Takeaways

- **Implication** $P \to Q$: false only when P is true and Q is false
- **Contrapositive**: $P \to Q \equiv \neg Q \to \neg P$ (always equivalent)
- **De Morgan's Laws**: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ and $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- **Implication law**: $P \to Q \equiv \neg P \vee Q$

Common Mistakes

- **Converse error**: $P \to Q$ does NOT mean $Q \to P$
- **Inverse error**: $P \to Q$ does NOT mean $\neg P \to \neg Q$
- **“Vacuously true”**: When P is false, $P \to Q$ is true regardless of Q

Practice Problems

1. Write the contrapositive of: "If today is Saturday, then there is no school."
2. Use the implication law to rewrite $P \to Q$ without using $\to$.
3. Given: If I study, I pass. If I pass, I graduate. I did not graduate. What can you conclude?
4. Is the statement "If $2 + 2 = 5$, then pigs can fly" true or false? Explain using the truth table for implication.
5. A parent says: "If you finish your vegetables, then you can have dessert."
   (a) You finish your vegetables and get dessert. Was the promise kept?
   (b) You don't finish vegetables but still get dessert. Was the promise broken?
   (c) Write the contrapositive of the parent's statement.
6. Use De Morgan's law to simplify $\neg(\neg A \vee B)$. Then verify your answer with a truth table.
7. On the Island of Knights and Knaves, A says: "If I am a Knight, then B is a Knight." What can you conclude if you know A is a Knight? What if A is a Knave? Then use the implication law to rewrite A's statement with "or." Have you seen this puzzle before?
8. Write the converse, inverse, and contrapositive of "If a number is divisible by 4, then it is even." Which of the four statements are true?
9. Build the truth table for $P \to \neg Q$.
10. Use a truth table to show that $\neg(P \to Q) \equiv P \wedge \neg Q$.
11. Write the contrapositive of "If it is not sunny, then we stay inside."
12. Simplify $\neg(P \vee \neg Q)$ using De Morgan's law, and write the result with no outer negation.
13. Is "If $x > 5$, then $x > 3$" true? Is its converse true? Explain.
14. Build the truth table for $(P \to Q) \wedge (Q \to P)$. (This is $P \leftrightarrow Q$, "P if and only if Q.")
15. A detective reasons: "If the suspect is guilty, then he was at the scene. If he was at the scene, then he was caught on camera. He was *not* caught on camera." Is the suspect guilty?
16. On the Island of Knights and Knaves, A says: "If B is a Knight, then I am a Knave." What are A and B? (Rewrite A's statement with "or" — does it remind you of a puzzle from Handout 3?)
17. Use a truth table to decide whether $P \to (Q \to R)$ is equivalent to $(P \wedge Q) \to R$.
18. Translate into logical symbols: "If it is neither raining nor cold, then we go to the park." (Let R, C, P stand for "it is raining," "it is cold," "we go to the park.") Then write the "if" part a second way, using De Morgan's law.

Practice Problems: Solutions

1. Contrapositive: **"If there is school, then today is not Saturday."** ("Not no school" is just "there is school.")
2. $P \to Q \equiv \neg P \vee Q$. So: **"Either P is false, or Q is true" (or both)**.
3. Using contrapositive chain:
   - ¬Graduate ⇒ ¬Pass (contrapositive of "Pass → Graduate")
   - ¬Pass ⇒ ¬Study (contrapositive of "Study → Pass")

   Conclusion: I did not study.
4. **True!** This is $P \to Q$ where P = "$2 + 2 = 5$" (false) and Q = "pigs can fly". When P is false, the implication $P \to Q$ is always true, regardless of Q. This is called "vacuously true."
5. (a) **Yes**, the promise was kept. (T → T = T)
   (b) **No**, the promise was NOT broken. The parent only promised what happens IF you finish vegetables. (F → T = T)
   (c) **"If you don't get dessert, then you didn't finish your vegetables."**
6. By De Morgan's law, $\neg(\neg A \vee B) \equiv \neg(\neg A) \wedge \neg B \equiv \mathbf{A} \wedge \neg\mathbf{B}$.

A	B	$\neg A \vee B$	$\neg(\neg A \vee B)$	$\neg B$	$A \wedge \neg B$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

   Columns 4 and 6 match. ✓
7. If A is a **Knight**: The statement "If I am a Knight, then B is a Knight" is true. Since A IS a Knight, the "if" part is true, so the "then" part must be true too. **B is a Knight.**

   If A is a **Knave**: The statement is false. For an implication to be false, we need T → F. But A is a Knave, so "I am a Knight" is false. The implication F → ? is always true, but a Knave's statement must be false. Contradiction! **A cannot be a Knave.**

   By the implication law, A's statement means "I am a Knave, or B is a Knight" — this is exactly Bridget and Connor from Handout 3 ("Either I am a knave or Connor is a knight"), and the answer is the same: both are Knights. It is also Handout 4's puzzle "B is a Knight, or I am a Knave."
8. Original: "If divisible by 4, then even" — **true**.
   - **Converse:** "If even, then divisible by 4" — **false** (e.g. 6).
   - **Inverse:** "If not divisible by 4, then not even" — **false** (e.g. 6).
   - **Contrapositive:** "If not even, then not divisible by 4" — **true**.
9.

P	Q	$\neg Q$	$P \to \neg Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

10. Both columns are F, T, F, F, so they are equivalent:

P	Q	$P \to Q$	$\neg(P \to Q)$	$P \wedge \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

11. Contrapositive: **"If we do not stay inside, then it is sunny."**

12. $\neg(P \vee \neg Q) \equiv \neg P \wedge \neg(\neg Q) \equiv \neg P \wedge Q$.

13. "If $x > 5$, then $x > 3$" is **true** (every number bigger than 5 is also bigger than 3). Its converse "If $x > 3$, then $x > 5$" is **false** — for example $x = 4$ is greater than 3 but not greater than 5.

14.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

It is true exactly when P and Q have the *same* truth value.

15. Guilty $\to$ Scene $\to$ Camera. He was not caught on camera, so by contrapositive: $\neg$Camera $\Rightarrow$ $\neg$Scene $\Rightarrow$ $\neg$Guilty. **No, the suspect is not guilty.**

16. If A is a Knight, his statement "$K_B \to \neg K_A$" is true; since A is a Knight, the "then" part $\neg K_A$ is false, so the "if" part must be false too — B is a Knave. If A were a Knave, the statement would be false, requiring K_B true and $\neg K_A$ false (K_A true) — impossible for a Knave. **A = Knight, B = Knave.** (By the implication law, A's statement means "B is a Knave or I am a Knave," that is, "at least one of us is a Knave" — a Handout 3 puzzle in disguise.)

17. Both are false only when $P = T$, $Q = T$, $R = F$, and agree in all 8 rows, so **yes, they are equivalent**:

P	Q	R	$Q \to R$	$P \to (Q \to R)$	$P \wedge Q$	$(P \wedge Q) \to R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

18. $\neg(R \vee C) \to P$. By De Morgan's law, the "if" part $\neg(R \vee C)$ is the same as $\neg R \wedge \neg C$, so the statement can also be written $(\neg R \wedge \neg C) \to P$.

Homework

1. If today is Thursday, then Jane's class goes to the library. If Jane's class goes to the library, then she brings home new library books. Today, Jane brought home no new library books. What can you conclude?

2. A teacher says: "If you do not take the final exam, you will get an F." Does this mean

   (a) If the student *does* take the final exam, they will *not* get an F?

   (b) If the student *does not* get an F, must they have taken the exam?

3. Many trucks carry the warning:

   > "If you do not see my mirrors, then I do not see you."

   Rewrite this message in an equivalent form **without** using the word "not."

4. Lewis Carroll puzzle:

   (a) All hummingbirds are richly colored.

   (b) No large birds live on honey.

   (c) Birds that do not live on honey are dull in color.

   What conclusion logically follows?

5. Babies, crocodiles, and logic (Lewis Carroll):

   (a) All babies are illogical.

   (b) Nobody is despised who can manage a crocodile.

   (c) Illogical persons are despised.

   What is the natural conclusion?

6. On the Island of Knights and Knaves, two inhabitants X and Y say:

   - X: "If Y is a knave, then I am a knave."
   - Y: "If X is a knave, then I am a knave."

   Determine who is a knight and who is a knave.

7. Carl says:

   > "If Sue is a knave, then this road leads to the capital."

   Later you learn that Carl himself is a knave. What can you conclude about the road?

8. M On the neighboring island, there are three kinds of people: **knights** (always true), **knaves** (always false), **normals** (sometimes true, sometimes false). Among A, B, and C, one is a knight, one is a knave, and one is normal. They say:

   - A: "B is normal."
   - B: "C is a knave."
   - C: "If I am a knight, then A is normal."

   Determine who is who.

9. M Back on the Island of Knights and Knaves (no Normals!), you meet three inhabitants A, B, and C. A says:

   "If I am a knight, then B and C are both knaves."

   What are A, B, and C?

10. M On the Island of Knights and Knaves, A says:

    "We are of the same type if and only if B is a Knight."

    What are A and B? Find every possible assignment.

11. Verify using a truth table that

$$A \to B \;\equiv\; \neg(A \wedge \neg B).$$

    Explain in words why this makes sense.

12. Write truth tables for the following formulas. Are they equivalent?

    (a) $(A \vee B) \wedge (A \vee C)$

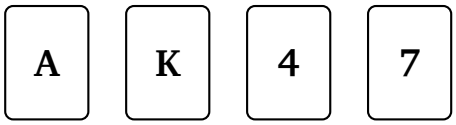
    (b) $A \vee (B \wedge C)$

13. Show that

$$(P \to Q) \vee (Q \to P)$$

    is always true. Explain in words why at least one of the two implications must hold.

14. **The Four-Card Puzzle.** Each card has a letter on one side and a number on the other. Four cards lie on the table:



    The rule is: "*If a card has a vowel on one side, then it has an even number on the other side.*" Which cards do you *have* to turn over to check whether the rule is being followed? Turn over as few as possible, and explain why each of the other cards doesn't matter.

    (When this puzzle is given to adults, most of them get it wrong!)

Quick Check Answers

1. **True.** When P is false, $P \to Q$ is always true (regardless of Q).
2. **You carry an umbrella.** The implication is true and P is true, so Q must be true.
3. **It is not raining.** By contrapositive: $\neg Q \Rightarrow \neg P$.
4. **No.** $P \to Q$ and $Q \to P$ are different (converse, not equivalent).
5. "If school is not canceled, then it doesn't snow."
6. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ (both false).
7. **No.** The inverse $\neg P \to \neg Q$ is NOT equivalent to $P \to Q$. (The contrapositive is equivalent, the inverse is not!)