

MATH 6

LOGIC II: LOGICAL OPERATIONS — SHORT 2026–27

Here is a strange and wonderful idea: we can calculate with *true* and *false* almost the way we calculate with numbers. Instead of $+$ and $\times$, we use three little words — NOT, AND, OR. In the 1800s **George Boole** turned this into an algebra, and about a hundred years later **Claude Shannon** noticed that the same algebra describes electrical switches: ON = true, OFF = false. Every computer you have ever used runs on Boole’s idea.

Statements That Are True or False

A **proposition** is a sentence that is either true (T) or false (F). “ $3 + 5 = 8$ ” is a true proposition; “Paris is the capital of Spain” is a false one. But “Close the door!” and “Is 7 prime?” are not propositions — a command or a question can’t be true or false. And “ $x + 3 = 10$ ” isn’t one *yet*: it depends on x . We name propositions with capital letters P , Q , R .

NOT, AND, OR

P	$\neg P$	P	Q	$P \wedge Q$	$P \vee Q$
T	F	T	T	T	T
T	F	T	F	F	T
F	T	F	T	F	T
F	T	F	F	F	F

- **NOT** ($\neg P$) flips the truth value: if P = “it is raining,” then $\neg P$ = “it is not raining.”
- **AND** ($P \wedge Q$) is true only when *both* parts are true. “I have a pencil and an eraser” is false if either one is missing.
- **OR** ($P \vee Q$) is true when *at least one* part is true.

Careful with “or”

In everyday talk, “or” sometimes means “one or the other, but not both.” In logic, OR always *includes* both: “It is cold or it is windy” is true on a cold, windy day too.

Truth Tables: Never Miss a Case

A **truth table** lists *every* combination of T and F for the basic statements. With 2 letters there are 4 rows; with 3 letters, 8 rows; with n letters, 2^n rows. To list them all, let the first column change every half, the next column every quarter, and so on, down to the last column, which changes every row.

Example: who can join my German class? I teach German, and I also speak French and Spanish. A student who already knows German doesn’t need my class. And a beginner can’t follow explanations in German yet, so I’ll explain in French or Spanish — the student must understand at least one of them. With F = “speaks French,” S = “speaks Spanish,” G = “speaks German,” I am looking for students with

$$(F \vee S) \wedge \neg G \quad (\text{“French OR Spanish, AND NOT German”}).$$

F	S	G	$F \vee S$	$\neg G$	$(F \vee S) \wedge \neg G$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	F	T	F

Three kinds of students fit: French and Spanish, only French, or only Spanish — as long as they don't speak German. The last row is the sad one: a student who speaks none of the three would love to learn German, but we'd have no language to talk in!

Knights and Knaves, by Truth Table

Back on the island: Knights always tell the truth, Knaves always lie. Let K_A mean "A is a Knight"; then "A is a Knave" is $\neg K_A$. A truth table lets us check *every* possibility at once. A row is **consistent** when each Knight's statement is true and each Knave's statement is false.

Example. Alice says: "We are both Knaves." Her statement is $S = \neg K_A \wedge \neg K_B$.

K_A	K_B	$S = \neg K_A \wedge \neg K_B$	Consistent?
T	T	F	✗ a Knight saying something false
T	F	F	✗ a Knight saying something false
F	T	F	✓ a Knave saying something false
F	F	T	✗ a Knave saying something true

Only one row survives: **Alice is a Knave and Bob is a Knight**. If *no* row survives, the situation is impossible; if several survive, the puzzle has several answers.

Common Mistakes

- **Treating OR as "only one."** $P \vee Q$ is true when both are true.
- **Missing rows.** Three letters need all 8 rows — use the halving pattern to list them.
- **Forgetting the speaker.** In Knights-and-Knives tables, compare each statement with *who* says it.

Homework

1. Construct the truth table for $\neg P \vee Q$.
2. Use truth tables to decide whether $\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$.
3. Let P = “it is cold” and Q = “it is raining.” Write in symbols: (a) it is cold but not raining; (b) it is neither cold nor raining.
4. A says, “B is a Knave or C is a Knave.” B says, “A is a Knight.” C says, “B is a Knave.” What are A, B, and C?
5. A says, “B is a Knight and C is a Knave.” B says, “C is a Knight.” C says, “A is a Knave.” What are A, B, and C?
6. A says, “Both of us are Knaves.” B says, “At least one of us is a Knight.” What are A and B?
7. On the Island of Knights and Knaves, you meet three inhabitants: Bob, Mel, and Peggy.
 - Bob says: “It is not true that Peggy is a knave.”
 - Mel says: “Peggy is a knight, or Bob is a knave.”
 - Peggy says: “I am a knight, and Bob is a knave.”

Determine who is a knight and who is a knave.

8. **M** On the Island of Knights and Knaves, you meet three inhabitants: Bozo, Carl, and Joe.
 - Bozo says: “Carl is a knave.”
 - Carl says: “Between Joe and me, exactly one of us is a knight.”
 - Joe says: “Bozo and I are of different kinds.”

Determine the status of each inhabitant.

9. **M** A traveler meets five islanders, A_1, A_2, A_3, A_4, A_5 . Each makes a claim about how many of the five are Knights:
 - A_1 says: “Exactly one of us is a Knight.”
 - A_2 says: “Exactly two of us are Knights.”
 - A_3 says: “Exactly three of us are Knights.”
 - A_4 says: “Exactly four of us are Knights.”
 - A_5 says: “All five of us are Knights.”

Who is a Knight and who is a Knave?

10. A long time ago, a king decided to test a knight who wished to marry one of his daughters. The king placed the knight in front of three doors. Behind one door was a princess. Behind another was a hungry tiger. The third room was empty.

The king also placed signs on the doors with the following rules:

- The sign on the **princess's** door is true.
- The sign on the **tiger's** door is false.
- The sign on the **empty** room may be either true or false.

The signs read:

Room I	Room II	Room III
Room III is empty	The tiger is in Room I	This room is empty

Which door should the knight open?

11. A political convention included exactly 100 politicians. Each politician was either crooked or honest. You are told:

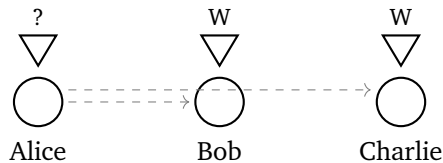
- At least one politician was honest.
- For any pair of politicians, at least one of them was crooked.

From these facts alone, can you determine how many honest and how many crooked politicians there were?

12. **H The Three Hats.** Three people—Alice, Bob, and Charlie—can see one another's hats but not their own. Each hat is either white or black. They are told:

- At least one person has a white hat.
- In each round, anyone who can logically deduce their own hat color should raise a hand. Everyone sees who raises a hand.

In round 1, nobody raises a hand. In round 2, again nobody raises a hand. At the start of round 3, Alice is the first to raise her hand and correctly says that her hat is white. Given that all three hats are white, explain her reasoning.



13. A says, "It is not true that B and C are both Knights." B says, "A is a Knave." C says, "B is a Knight." What can you conclude about A, B, and C?