

MATH 6

LOGIC II: LOGICAL OPERATIONS — FULL 2026–27

A Brief History of Logic

Logic begins with a wonderfully strange idea: statements can be handled almost like numbers. Instead of adding and multiplying, we combine *true* and *false* using operations such as NOT, AND, and OR.

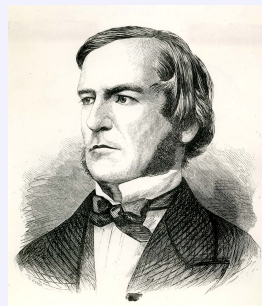
History of Logic

People have studied good reasoning for a very long time. More than **2,300 years ago**, the Greek philosopher **Aristotle** (384–322 BC) organized patterns of valid arguments. His key insight was that a good argument depends on its *structure*, not on whether it is about stars, animals, or sandwiches.

The great leap toward modern symbolic logic came in the 1800s. **George Boole** invented an algebra in which the variables could represent statements that are true or false. This was the beginning of what we now call *Boolean algebra*. **Augustus De Morgan** developed important rules for negating combinations of statements. We will discover one of those rules ourselves later in this handout.

At first this may have looked like a curious game with symbols. Then, in 1937, **Claude Shannon** showed that Boolean algebra could describe electrical switches: ON could represent true and OFF could represent false. That connection became one of the basic ideas behind digital circuits and computers.

The symbols we use today were standardized gradually; they are not all Boole's original notation. But the central idea is his: reasoning can be studied by careful algebraic rules. In this handout, we will begin with the three most useful operations: NOT, AND, and OR.



George Boole
(1815–1864)

Propositions and Truth Values

Before we can work with logical operations, we must understand the basic objects we are combining. In logic, the fundamental building blocks are called **propositions** — statements that are either *true* or *false*.

A proposition can be simple, such as “3 is an odd number,” or more complex, such as “It is raining in Boston at noon today.” What matters is that the sentence gives enough information to be definitely true or definitely false. Logic studies how these truth-valued statements behave and how they can be combined.

Proposition

A **proposition** is a statement that is either **true** (T) or **false** (F), but not both. Its **truth value** is T or F.

- True example: “ $3 + 5 = 8$.”
- False example: “Paris is the capital of Spain.”
- Not a proposition: “Close the door.” (This is a command.)
- Not a proposition: “Is 7 prime?” (This is a question.)
- Not yet a proposition: “ $x + 3 = 10$ ” (Its truth depends on x .)

We use capital letters P , Q , R to represent propositions.

Logical Operations

You already combine statements all the time. “It is *not* raining.” “I finished my homework *and* I cleaned my room.” “We’ll go to the park *or* the zoo.” The little words NOT, AND, OR build new statements out of old ones — and the truth of the new statement depends only on the truth of the old ones. In logic we give these words symbols and pin down exactly how they work.

NOT (negation)

If P is a proposition, then its negation is written $\neg P$. It says the opposite of P : if P = “It is raining,” then $\neg P$ = “It is not raining.” Exactly one of them is true.

$$\neg P = \text{“NOT } P\text{”}$$

P	$\neg P$
T	F
F	T

AND (conjunction)

The statement “ P AND Q ” is written $P \wedge Q$. It is true only if *both* P and Q are true. “I have a pencil and I have an eraser” is false if either one is missing.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

OR (disjunction)

The statement “ P OR Q ” is written $P \vee Q$.

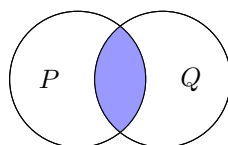
In logic, OR is **inclusive**: $P \vee Q$ is true if *at least one* of P or Q is true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

In everyday English, “or” sometimes means “one or the other, but not both.” In mathematics and logic, unless stated otherwise, “or” includes the possibility of both.

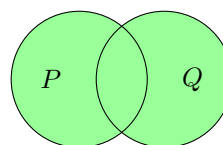
In the pictures below, each circle represents all the situations in which a statement is true. The shaded part shows the situations in which the combined statement is true. This is why intersection and union pictures are useful memory aids for AND and OR.

$P \wedge Q$ (AND)



Only the overlap

$P \vee Q$ (OR)



Everything in either

Quick Check

1. If P is true, what is $\neg P$?
2. If P is true and Q is false, what is $P \wedge Q$?
3. If P is false and Q is true, what is $P \vee Q$?
4. Is “ $3 > 2$ ” a proposition? What is its truth value?
5. Which of these are propositions: “ $x + 5 = 10$,” “Close the door,” and “7 is prime”? Explain.

Truth Tables

As statements get longer, it is easy to lose track of when they are true. A **truth table** lists *every* possible combination of truth values for the basic propositions, and shows the truth value of the compound statement in each case. Because nothing is left out, a truth table never misses a case.

How to build the rows. For two variables, use

P	Q
T	T
T	F
F	T
F	F

The first column changes every two rows; the second changes every row. For three variables, the columns change every four rows, every two rows, and every row. In general, n variables require 2^n rows.

Example. The truth table for $P \vee \neg Q$ is:

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Who can join my German class?

Suppose I teach German, and I also speak French and Spanish. I am looking for new students. A student who already knows German doesn't need my class. And since a beginner can't follow explanations in German yet, I will explain things in French or in Spanish — so the student must understand at least one of them.

For a given student, let

F = “speaks French,” S = “speaks Spanish,” G = “speaks German.”

I am looking for students for whom

$$(F \vee S) \wedge \neg G$$

is true: “French OR Spanish, AND NOT German.” With three variables we need $2^3 = 8$ rows — one for every kind of student. Build helper columns first, then combine them:

F	S	G	$F \vee S$	$\neg G$	$(F \vee S) \wedge \neg G$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	F	T	F

Exactly three kinds of students fit: those who speak both French and Spanish (but not German), those who speak only French, and those who speak only Spanish. Notice the last row: a student who speaks none of the three languages would love to learn German — but we would have no language to talk in!

Quick Check

- How many rows does a truth table need if it has 3 variables (P, Q, R)?
- In a truth table for $\neg P \wedge Q$, what is the result when P is false and Q is true?

Knights and Knaves Using Truth Tables

A **Knight** always tells the truth. A **Knave** always lies. We let K_A mean “A is a Knight”. Then “A is a Knave” is $\neg K_A$. Truth tables allow us to determine which assignments are logically possible.

The three examples below are puzzles you already met in Handout 3, where we solved them by trying cases. Here we solve them again with truth tables. A truth table is longer, but it has one great advantage: it lists *every* possibility, so we can never forget a case.

“We Are Both Knaves”

Alice says: “We are both knaves.”

Let K_A = “Alice is a Knight”, K_B = “Bob is a Knight”.

Alice’s statement is:

$$S = (\neg K_A) \wedge (\neg K_B).$$

Her identity must match the truth of her statement:

- If $K_A = \text{T}$, then S must be true.
- If $K_A = \text{F}$, then S must be false.

K_A	K_B	$\neg K_A$	$\neg K_B$	$S = (\neg K_A) \wedge (\neg K_B)$	Consistent?
T	T	F	F	F	✗ (Alice-Knight saying F)
T	F	F	T	F	✗ (Alice-Knight saying F)
F	T	T	F	F	✓ (Alice-Knave saying F)
F	F	T	T	T	✗ (Alice-Knave saying T)

Only one row works:

Alice is a Knave, Bob is a Knight.

Same or Different?

Alice says: “We are the same kind.” Bob says: “We are of different kinds.”

Let K_A and K_B mean “Alice is a Knight” and “Bob is a Knight.”

Alice’s statement means:

$$S_A = \text{same kind} = (K_A \wedge K_B) \vee (\neg K_A \wedge \neg K_B),$$

i.e. both Knights or both Knaves.

Bob’s statement means:

$$S_B = \text{different kind} = (K_A \wedge \neg K_B) \vee (\neg K_A \wedge K_B),$$

i.e. exactly one Knight and one Knave.

A Knight tells a true statement; a Knave tells a false statement.

We now check all possibilities with a truth table:

K_A	K_B	$\neg K_A$	$\neg K_B$	$S_A = \text{same}$	$S_B = \text{different}$	Consistent?
T	T	F	F	T	F	✗ (Bob–Knight saying F)
T	F	F	T	F	T	✗ (Alice–Knight saying F)
F	T	T	F	F	T	✓ (Alice–Knave F, Bob–Knight T)
F	F	T	T	T	F	✗ (Alice–Knave saying T)

Only one row is logically consistent:

Alice is a Knave and Bob is a Knight.

Claire and Desmond

One person is named Claire and the other is named Desmond. You do not know what type either of them is. Claire says:

“We are both Knights.”

Desmond then says:

“Either Claire is a Knight or I am a Knight, but we are not both Knights.”

What is Claire? Can we determine what Desmond is?

Let

$$K_C = \text{“Claire is a Knight”}, \quad K_D = \text{“Desmond is a Knight”}.$$

Claire’s statement is

$$S_C = K_C \wedge K_D$$

(“We are both Knights.”)

Desmond’s statement says: *at least one of us is a Knight, but not both*. That is the *exclusive or*:

$$S_D = (K_C \vee K_D) \wedge \neg(K_C \wedge K_D).$$

A Knight always makes a true statement, a Knave always makes a false one.

Now we make a truth table:

K_C	K_D	$S_C = \text{both knights}$	$S_D = \text{exactly one knight}$	Claire OK?	Desmond OK?
T	T	T	F	Knight saying T ✓	Knight saying F ✗
T	F	F	T	Knight saying F ✗	Knave saying T ✗
F	T	F	T	Knave saying F ✓	Knight saying T ✓
F	F	F	F	Knave saying F ✓	Knave saying F ✓

The only rows where *both* are consistent with being Knight/Knave are:

$$(K_C, K_D) = (F, T) \quad \text{and} \quad (F, F).$$

In both cases $K_C = F$, so:

Claire is a Knave.

Desmond is a Knight in one row and a Knave in the other, so his type *cannot* be determined:

Desmond could be either a Knight or a Knave.

Quick Check

8. When using truth tables for Knights and Knaves, a row is “consistent” if what condition is met?
9. If a puzzle has no consistent rows in the truth table, what does this mean?

Key Takeaways

- A **proposition** is a statement that is either true (T) or false (F)
- **NOT** ($\neg P$): flips the truth value
- **AND** ($P \wedge Q$): true only when *both* are true
- **OR** ($P \vee Q$): true when *at least one* is true (inclusive)
- A **truth table** lists all possible combinations of truth values
- For Knights/Knaves: Knight’s statement must be true; Knave’s must be false

Common Mistakes

- **Confusing inclusive/exclusive OR**: In math, OR includes “both true”
- **Forgetting rows**: With n variables, need 2^n rows in truth table
- **Wrong negation**: $\neg(P \wedge Q) \not\equiv \neg P \wedge \neg Q$
- **Rushing consistency check**: Verify BOTH speakers match their type

Practice Problems

1. Construct the truth table for $(P \wedge Q) \vee \neg P$.
2. Construct the truth table for $\neg(P \wedge Q)$.
3. Construct the truth table for $(\neg P) \vee (\neg Q)$.
4. Compare your answers to problems 2 and 3. What do you notice?
5. Determine whether $P \vee (Q \wedge \neg P)$ is equivalent to $P \vee Q$.
6. Solve using a truth table: A says, "At least one of us is a knight." B says nothing.
7. A says, "B is a Knight, or I am a Knave." B says, "A is a Knight." What are A and B?
8. Construct the truth table for $P \vee (Q \wedge R)$. (Three variables — how many rows?)
9. Construct the truth table for $\neg P \wedge \neg Q$.
10. Use a truth table to decide whether $\neg(\neg P)$ is equivalent to P .
11. Let P = "the number is even" and Q = "the number is positive." Write in symbols: (a) the number is even and positive; (b) the number is not even.
12. Which of the following are propositions? For each proposition, give its truth value.
 - (a) "7 is a prime number."
 - (b) "Close the door."
 - (c) " $2x = 6$."
 - (d) "The Moon is made of cheese."
13. A says, "I am a Knight." Using a truth table, what can you conclude about A?
14. A says, "I am a Knight or B is a Knave." B says nothing. Use a truth table to find every possibility.
15. A says, "At least one of us is a Knave." B says, "At least one of us is a Knight." What are A and B?
16. Three inhabitants: A says, "C is a Knight." B says, "A is a Knight." C says, "I am a Knight." What are A, B, and C?
17. A says, "I am a Knight and B is a Knave." B says nothing. Use a truth table to find every possibility. Compare with problem 14.
18. Use a truth table to decide whether $(P \vee Q) \wedge \neg Q$ is equivalent to $P \wedge \neg Q$.
19. A says, "I am a Knight and B is a Knight." B says, "A is a Knave." What are A and B?
20. Three inhabitants: A says, "B is a Knave." B says, "C is a Knave." C says, "A is a Knight." What are A, B, and C?

Practice Problems: Solutions

1. Truth table for $(P \wedge Q) \vee \neg P$:

P	Q	$P \wedge Q$	$\neg P$	$(P \wedge Q) \vee \neg P$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

2. Truth table for $\neg(P \wedge Q)$:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

3. Truth table for $(\neg P) \vee (\neg Q)$:

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

4. **They are identical!** This is one of De Morgan's Laws: $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.
(You'll learn more about De Morgan's Laws in the next handout.)

5. Truth tables for both expressions:

P	Q	$\neg P$	$Q \wedge \neg P$	$P \vee (Q \wedge \neg P)$	P	Q	$P \vee Q$
T	T	F	F	T	T	T	T
T	F	F	F	T	T	F	T
F	T	T	T	T	F	T	T
F	F	T	F	F	F	F	F

Yes, they are equivalent (same final columns).

6. A's statement: $S = K_A \vee K_B$ ("at least one knight")

K_A	K_B	S	Consistent?
T	T	T	✓
T	F	T	✓
F	T	T	✗ (Knave saying T)
F	F	F	✓

Three possibilities: (A=Knight, B=Knight), (A=Knight, B=Knave), or (A=Knave, B=Knave).

7. A's statement is $S_A = K_B \vee \neg K_A$. If A were a Knave, the part "I am a Knave" would be true, so the whole OR-statement would be true — but a Knave cannot make a true statement. So A is a Knight. Then "I am a Knave" is false, so the other part, "B is a Knight," must be true. Check B: a Knight truly says "A is a Knight" ✓. **A = Knight, B = Knight.**
8. Three variables need $2^3 = 8$ rows:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

9. 10. $\neg(\neg P)$ has the same column as P ($T \rightarrow F \rightarrow T$ and $F \rightarrow T \rightarrow F$), so **yes**, $\neg(\neg P)$ is equivalent to P (double negation).
11. (a) $P \wedge Q$. (b) $\neg P$.
12. (a) Proposition, **True**. (b) **Not** a proposition (a command has no truth value). (c) **Not** a proposition (its truth depends on x). (d) Proposition, **False**.
13. Both rows are consistent: if A is a Knight the statement “I am a Knight” is true, and if A is a Knave it is false. **We cannot determine A’s type**.
14. A’s statement is $S = K_A \vee \neg K_B$.

K_A	K_B	$S = K_A \vee \neg K_B$	Consistent?
T	T	T	✓ (Knight saying T)
T	F	T	✓ (Knight saying T)
F	T	F	✓ (Knave saying F)
F	F	T	✗ (Knave saying T)

Three possibilities: (Knight, Knight), (Knight, Knave), (Knave, Knight). The only thing we learn is that they are *not both* Knaves.

15. If A were a Knave, then “at least one of us is a Knave” would be true (A himself is one) — a Knave telling the truth. So A must be a Knight. Then his statement is true, so B is a Knave. But B then says “at least one of us is a Knight,” which is true (A is one) — a Knave telling the truth. Every case fails: **no consistent assignment exists**.
16. If A is a Knight: “C is a Knight” is true, so C is a Knight; B truly says “A is a Knight,” so B is a Knight; C truly says “I am a Knight” ✓ — **all three Knights**. If A is a Knave: “C is a Knight” is false, so C is a Knave; B falsely says “A is a Knight,” so B is a Knave; C (Knave) falsely says “I am a Knight” ✓ — **all three Knaves**. **Two solutions: all Knights, or all Knaves**.
17. A’s statement is $S = K_A \wedge \neg K_B$.

K_A	K_B	$S = K_A \wedge \neg K_B$	Consistent?
T	T	F	✗ (Knight saying F)
T	F	T	✓ (Knight saying T)
F	T	F	✓ (Knave saying F)
F	F	F	✓ (Knave saying F)

Three possibilities: (Knight, Knave), (Knave, Knight), (Knave, Knave). Compared with problem 14, changing “or” to “and” changes which row is ruled out.

18. Both give the column F, T, F, F , so **yes, they are equivalent:**

P	Q	$\neg Q$	$(P \vee Q) \wedge \neg Q$	$P \wedge \neg Q$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

19. If A is a Knight, “I am a Knight and B is a Knight” is true, so B is a Knight; but then B’s “A is a Knave” is false — a Knight lying, contradiction. So A is a Knave, and his statement is false (fine, since he is not a Knight). B’s “A is a Knave” is then true, so B is a Knight. **A = Knave, B = Knight.**

20. If A is a Knight: B is a Knave (A’s statement); B (Knave) falsely says “C is a Knave,” so C is a Knight; C truly says “A is a Knight” ✓ — (Knight, Knave, Knight). If A is a Knave: B is a Knight; B truly says “C is a Knave,” so C is a Knave; C (Knave) falsely says “A is a Knight” ✓ — (Knave, Knight, Knave). **Two solutions: (Knight, Knave, Knight) or (Knave, Knight, Knave).**

Homework

1. Construct the truth table for $\neg P \vee Q$.
2. Use truth tables to decide whether $\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$.
3. Let P = “it is cold” and Q = “it is raining.” Write in symbols: (a) it is cold but not raining; (b) it is neither cold nor raining.
4. A says, “B is a Knave or C is a Knave.” B says, “A is a Knight.” C says, “B is a Knave.” What are A, B, and C?
5. A says, “B is a Knight and C is a Knave.” B says, “C is a Knight.” C says, “A is a Knave.” What are A, B, and C?
6. A says, “Both of us are Knaves.” B says, “At least one of us is a Knight.” What are A and B?
7. On the Island of Knights and Knaves, you meet three inhabitants: Bob, Mel, and Peggy.
 - Bob says: “It is not true that Peggy is a knave.”
 - Mel says: “Peggy is a knight, or Bob is a knave.”
 - Peggy says: “I am a knight, and Bob is a knave.”

Determine who is a knight and who is a knave.

8. **M** On the Island of Knights and Knaves, you meet three inhabitants: Bozo, Carl, and Joe.
 - Bozo says: “Carl is a knave.”
 - Carl says: “Between Joe and me, exactly one of us is a knight.”
 - Joe says: “Bozo and I are of different kinds.”

Determine the status of each inhabitant.

9. **M** A traveler meets five islanders, A_1, A_2, A_3, A_4, A_5 . Each makes a claim about how many of the five are Knights:
 - A_1 says: “Exactly one of us is a Knight.”
 - A_2 says: “Exactly two of us are Knights.”
 - A_3 says: “Exactly three of us are Knights.”
 - A_4 says: “Exactly four of us are Knights.”
 - A_5 says: “All five of us are Knights.”

Who is a Knight and who is a Knave?

10. A long time ago, a king decided to test a knight who wished to marry one of his daughters. The king placed the knight in front of three doors. Behind one door was a princess. Behind another was a hungry tiger. The third room was empty.

The king also placed signs on the doors with the following rules:

- The sign on the **princess's** door is true.
- The sign on the **tiger's** door is false.
- The sign on the **empty** room may be either true or false.

The signs read:

Room I	Room II	Room III
Room III is empty	The tiger is in Room I	This room is empty

Which door should the knight open?

11. A political convention included exactly 100 politicians. Each politician was either crooked or honest. You are told:

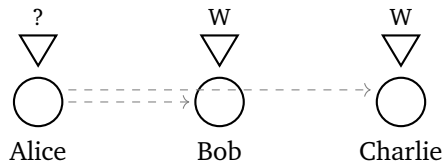
- At least one politician was honest.
- For any pair of politicians, at least one of them was crooked.

From these facts alone, can you determine how many honest and how many crooked politicians there were?

12. **H The Three Hats.** Three people—Alice, Bob, and Charlie—can see one another’s hats but not their own. Each hat is either white or black. They are told:

- At least one person has a white hat.
- In each round, anyone who can logically deduce their own hat color should raise a hand. Everyone sees who raises a hand.

In round 1, nobody raises a hand. In round 2, again nobody raises a hand. At the start of round 3, Alice is the first to raise her hand and correctly says that her hat is white. Given that all three hats are white, explain her reasoning.



13. A says, “It is not true that B and C are both Knights.” B says, “A is a Knave.” C says, “B is a Knight.” What can you conclude about A, B, and C?

Quick Check Answers

1. False.
2. False (AND requires both to be true).
3. True (OR requires at least one to be true).
4. Yes, it is a proposition. Its truth value is True.
5. Only “7 is prime” is a proposition (and it is true). The first sentence depends on the unknown value of x , and the second is a command rather than a statement that can be true or false.
6. $2^3 = 8$ rows (one for each combination of T/F for the three variables).
7. **True.** Since P is false, $\neg P$ is true. And Q is true. So $\neg P \wedge Q = T \wedge T = T$.
8. Each person’s statement must match their type: Knights must make true statements, Knaves must make false statements.
9. The puzzle has no solution under the Knights-and-Knaves rules: no assignment makes every speaker’s statement match the speaker’s type.