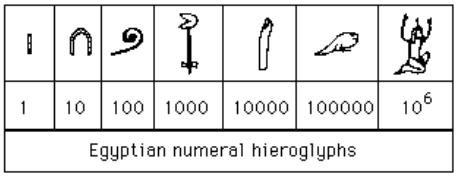
Math 5b. Class work 4.

Numeral systems.

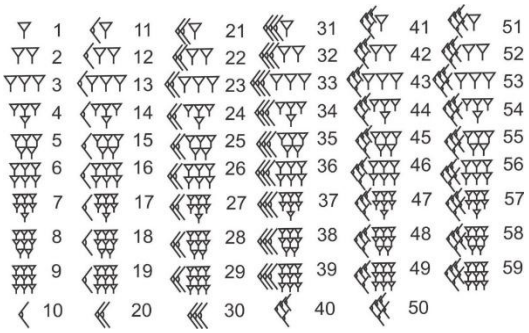
Over the course of human history, many different numeral systems have appeared in different cultures. The earliest numeral systems were not place-value systems.
A good example of such a system is the ancient Egyptian decimal numeral system.



Egyptian numeral hieroglyphs

It is a decimal system, the symbols can be written in almost any order, but usually were read from right to left and/or from bottom to top.

1,333,330 in Egyptian hieroglyphs from the Edfu Temple (237–57 BCE) in Egypt.



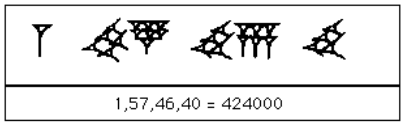
The Babylonian numeral system (from about 2000 BC) used only two symbols to represent any number from 1 to 59. It was based on 60, but its symbols reflected groups of ten.

Also, it was a position value system, for example, the number 62 was shown as , which means one 60 and 2 ones. The use of this sexagesimal (base-60) system is still noticeable today. We still have 60 minutes in an hour, 60 seconds in a minute, and 360° in a full circle. Now we adopt a notation where we separate the numerals by commas when we use our digits to represent Babylonian notation, for example, 1,57,46,40 represents the sexagesimal number

$$60^3 \cdot 1 + 60^2 \cdot 57 + 60^1 \cdot 46 + 60^0 \cdot 40$$

which, in decimal notation is 424000.

Babylonian mathematical tablet Plimpton 322.Credit...Christine Proust and Columbia University



Here is **1,57,46,40 in Babylonian numerals** https://mathshistory.st-andrews.ac.uk/HistTopics/Babylonian_mathematics/

Another very well-known numeral system is roman; it was used for thousands of years and in some cases is still used now as well. It is a decimal (base-10) system, but the symbols (letters) are used in an unusual way.

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

For example, 4 is one less than 5, so 4 can be written as IV. Same principle of subtractive notation is used for 9 -> IX, 40 and 90 -> XL and XC, 400 and 900 -> CD and CM
Some other examples:

- 29 = XX + IX = **XXIX**.
- 347 = CCC + XL + VII = **CCCXLVII**.
- 789 = DCC + LXXX + IX = **DCCLXXXIX**.
- 2,421 = MM + CD + XX + I = **MMCDXXI**

Any missing place (represented by a zero in the place-value equivalent) is omitted, as in Latin (and English) speech:

- 160 = C + LX = **CLX**
- 207 = CC + VII = **CCVII**
- 1,009 = M + IX = **MIX**
- 1,066 = M + LX + VI = **MLXVI**

Can a non-decimal place-value system be created? For example, with base 5?
Let's see, how we can create this kind of system (we use our normal digits).

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_5	1	2	3	4	10	11	12	13	14	20

11	12	13	14	15	16	17	18	19	20
21	22	23	24	30	31	32	33	34	40

21	22	23	24	25	26	27	28	29	30
41	42	43	44	100	101	102	103	104	105

We only have 5 digits (0, 1, 2, 3, 4), and the first four natural numbers in such a system will be represented as one-digit numbers. Number 5 is then written as a 2-digit number, with first digit 1 (place value equal to 5^1) and 0 of "units". Any number is now written in the form

$$\ldots + 5^3 \cdot n + 5^2 \cdot m + 5^1 \cdot k + 5^0 \cdot p, \qquad n, m, k, p \quad \text{are} \quad 0, 1, 2, 3, 4$$

$33 = 25 + 5 + 3 = 5^2 \cdot 1 + 5^1 \cdot 1 + 5^0 \cdot 3 = 113_5$

$195 = 125 + 25 \cdot 2 + 5 \cdot 4 = 5^3 \cdot 1 + 5^2 \cdot 2 + 5^1 \cdot 4 + 0 = 1240_5$

And vice versa, we can convert a number from base 5 to decimal system:

$2312_5 = 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 1 + 2 = 250 + 75 + 5 + 2 = 332$

Let's add the following two base-5 numbers: 1240_5 and 2312_5 (I omitted $_5$ notation):

$$\begin{array}{r} 1240 \\ +\ \underline{2312} \\ 4102 \end{array}$$

$4102_5 = 5^3 \cdot 4 + 5^2 \cdot 1 + 5^1 \cdot 0 + 2 = 125 \cdot 4 + 25 \cdot 1 + 0 + 2 = 500 + 25 + 2 = 527$

Let' s try to introduce a new digit S for 10 and then create an based-11 system.

$11^2 = 121, \qquad 11^3 = 1331$

$890 = 121 \cdot 7 + 11 \cdot 3 + 10 = 11^2 \cdot 7 + 11^1 \cdot 3 + 10 = 73\mathsf{S}_{11}$

$4\mathsf{S}2_{11} = 11^2 \cdot 4 + 11^1 \cdot 10 + 2 = 484 + 110 + 2 = 596$

Another important place value system is the binary (base-2) system which uses only two digits, 0, and 1.

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_2	1	10	11	100	101	110	111	1000	1001	1010

In this system place value of a digit is a power of 2:

$$\ldots + 2^3 \cdot (0,1) + 2^2 \cdot (0,1) + 2^1 \cdot (0,1) + 2^0 \cdot (0,1)$$

$11 = 8 + 2 + 1 = 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1011_2$

$75 = 64 + 8 + 2 + 1 = 2^6 \cdot 1 + 2^5 \cdot 0 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1001011_2$

$189 = 128 + 32 + 16 + 8 + 4 + 1$

$= 2^7 \cdot 1 + 2^6 \cdot 0 + 2^5 \cdot 1 + 2^4 \cdot 1 + 2^3 \cdot 1 + 2^2 \cdot 1 + 2^1 \cdot 0 + 2^0 \cdot 1$

$= 10111101_2$

Binary numbers can be converted to decimal numbers. We will do it from right to left for convenience.

$$11010111_2 = 2^0 \cdot 1 + 2^1 \cdot 1 + 2^2 \cdot 1 + 2^3 \cdot 0 + 2^4 \cdot 1 + 2^5 \cdot 0 + 2^6 \cdot 1 + 2^7 \cdot 1$$
$$= 128 + 64 + 16 + 4 + 2 + 1 = 215$$

There is an easy way to write a decimal number in any place value system. Let's do it for number 736, and convert it into a base 5 system:

$$736_{10} \to ????_5$$
$$736:5 = 147R1$$
$$147:5 = 29R2$$
$$29:5 = 5R4$$
$$5:5 = 1R0$$
$$1:5 = 0R1$$

Now, if we write all the remainders in reverse order, 10421_5, we will get the same number written in the base 5 system. Let' check it with our previous algorithm and understand why it works.

$$736_{10} = 625 + 111 = 5^4 \cdot 1 + 111 = 5^4 \cdot 1 + 125 \cdot 0 + 111 = 5^4 \cdot 1 + 5^3 \cdot 0 + 111$$
$$= 5^4 \cdot 1 + 5^3 \cdot 0 + 25 \cdot 4 + 11 = 5^4 \cdot 1 + 5^3 \cdot 0 + 5^2 \cdot 4 + 10 + 1 =$$
$$= 5^4 \cdot 1 + 5^3 \cdot 0 + 5^2 \cdot 4 + 5^1 \cdot 2 + 5^0 \cdot 1 = 10421_5$$

From the expression $5^4 \cdot 1 + 5^3 \cdot 0 + 5^2 \cdot 4 + 5^1 \cdot 2 + 5^0 \cdot 1$ we can factor out the factor 5:

$$5^4 \cdot 1 + 5^3 \cdot 0 + 5^2 \cdot 4 + 5^1 \cdot 2 + 5^0 \cdot 1 = 5^0(5^1(5^3 \cdot 1 + 5^2 \cdot 0 + 5^1 \cdot 4 + 2) + 1) = \cdots =$$
$$= 5^0(5^1(5^1(5^1(5^1 \cdot 1 + 0) + 4) + 2) + 1)$$

Problem solving examples:

Example 1.

Express the number 342 in a ternary (3-based) system:

One way to do it:

In the ternary system we only have 0, 1, 2 as digits. Power of 3 table:

3^0	3^1	3^2	3^3	3^4	3^5
1	3	9	27	81	243

$$243 < 342 < 243 \cdot 2$$

So, our highest place value is 3^5 (243)

$$342 = 243 + 99 = 3^5 \cdot 1 + 81 + 18 = 3^5 \cdot 1 + 3^4 \cdot 1 + 3^2 \cdot 2 + 0$$

But, keep in mind, we need all power of 3, from 5^{th} to 0 to be present.

$$342 = 3^5 \cdot 1 + 3^4 \cdot 1 + 3^3 \cdot 0 + 3^2 \cdot 2 + 3^1 \cdot 0 + 3^0 \cdot 0 = 110200_3$$

This solution shows the way how placed value system of writing numbers works.

Another way:

$$342_{10} \rightarrow ????_3$$
$$342:3 = 114R0$$
$$114:3 = 38R0$$
$$38:3 = 12R2$$
$$12:3 = 4R0$$
$$4:3 = 1R1$$
$$1:3 = 0R1$$

By writing the remainders in reverse order (110200) we are getting a right answer: 110200_3. This is the easy technical way to express any number in any system, based on a number smaller then 10. (But also, we have to understand why it works).

Example 2:

Express the number 2012_3 in a decimal (10-based) system.

Place values in the number are powers of 3. The last digit represents units, the second from the left has a place value equal to $3^1 = 3$, the next one has a place value equal to $3^2 = 9$ and so on.

$$2012_3 = 3^0 \cdot 2 + 3^1 \cdot 1 + 3^2 \cdot 0 + 3^3 \cdot 2$$

Or, written in a usual way:

$$2012_3 = 3^3 \cdot 2 + 3^2 \cdot 0 + 3^1 \cdot 1 + 3^0 \cdot 2 = 27 \cdot 2 + 0 + 3 + 2 = 54 + 3 + 2 = 59$$

Example 3.

A three-digit number starts with the digit 4. If the digit 4 is moved from the beginning to the end of the number, the new number is 0.75 times the original number. Find the original number.

This 3-digit number can be written as $\overline{4ab}$. The bar on the top of the number indicates, that it's not a product of 4, a and b, but a number, whose digits are 4, a, and b.

$$\overline{4ab} = 100 \cdot 4 + 10 \cdot a + b$$

If the digit 4 moved to the end of the number, a new number will look like this expression

$$\overline{ab4} = 100 \cdot a + 10 \cdot b + 4$$

This new number is exactly 0.75 times the original number.

$$\overline{ab4} = 0.75 \cdot \overline{4ab} = \frac{3}{4} \cdot \overline{4ab}$$

$$100 \cdot a + 10 \cdot b + 4 = \frac{3}{4} \cdot (100 \cdot 4 + 10 \cdot a + b)$$

If we multiply both sides of the equation by 4 and then use the distributive property, the equation will become:

$$4 \cdot (100 \cdot a + 10 \cdot b + 4) = 3 \cdot (100 \cdot 4 + 10 \cdot a + b)$$
$$400 \cdot a + 40 \cdot b + 16 = 300 \cdot 4 + 30 \cdot a + 3 \cdot b$$

Let's subtract $30a$ and $3b$, and also 16 from both side of the equation. We can do it; the equality will still be in place.

$$400a + 40b - 30a - 3b = 1200 - 16$$

$$370a + 37b = 1184$$

If we factor out the factor 37 in the left side:

$$37 \cdot (10a + b) = 1184$$

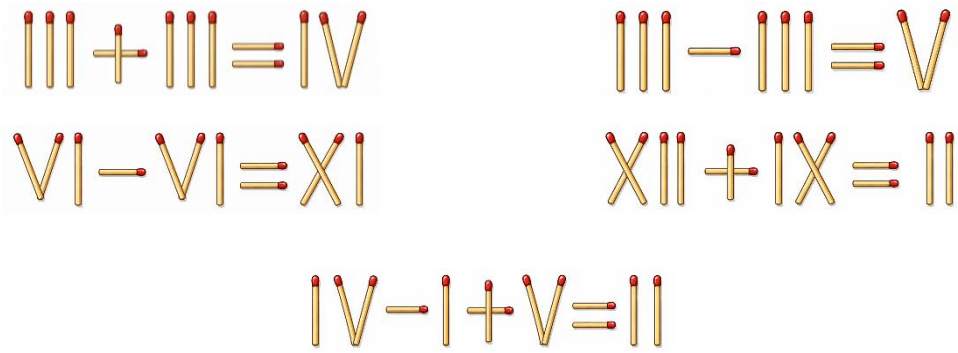
Divide both sides by 37:

$$10a + b = 32 = 10 \cdot 3 + 2$$

Answer: $a = 3,\ b = 2,$ initial number is 432.

Exercises:

1. Move one matchstick to make the equation correct.



2. Write as binary:
   $a.$ 1, 2, 3, 5, 7, 8, 10, 16, 25, 30, 32, 40, 55, 64, 100
   $b.$ 131, 178, 724, 1231,

3. Write as a decimal:
   $a.$ $1_2;$ $10_2;$ $11_2;$ $100_2;$ $101_2;$ $111_2;$ 1000_2
   $b.$ $101010_2;$ $111000_2;$ $100011_2;$ $110110_2;$

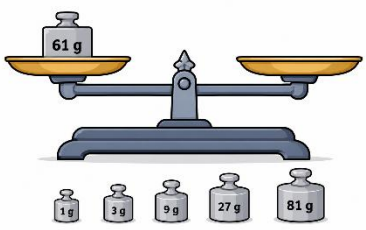
4. Append the same digit to both the left and the right of the number 10 so that the resulting four-digit number is divisible by 12. Find the digit.

5. The tens digit of a two-digit number is twice its ones digit. If the square of its tens digit is added to the number, the result is the square of an integer. Find the original two-digit number.

6. Write the numbers 245 and 324 in base-6 place-value system. Remember that in this system you will have only 0, 1, 2, 3. 4, and 5 as digits.

7. Write the numbers 234_6 and 403_6 written in the base 6 place-value system (small number 6 shows that the number is not in decimal, but in 6-based system) in the decimal system.

8. Michael chose a number between 1 and 1000, inclusive. Robert may ask only questions that can be answered with "yes" or "no" (and Michael always tells the truth). Can Robert determine the chosen number in 10 questions?



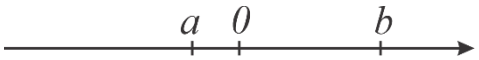
9. You have a balance scale with two pans and one weight of each of the following masses: 1 g, 3 g, 9 g, 27 g, and 81 g. How can you balance a 61 g object placed on one pan of the scale?

10. A natural number was multiplied successively by each of its digits. The final result was 1995. Find the original number. (Hint: do the prime factorization)

Review Problems.

11. Points a and b are marked on a coordinate line. Compare:

$a.\ a+b\ \ldots\ 0;$ $b.\ a-b\ \ldots\ 0;$ $c.\ ab\ \ldots\ 0;$ $d.\ \frac{b}{a}\ \ldots.\ 1$



12. Without doing calculation prove that

$a.\ \frac{1}{3}+\frac{1}{4}>\frac{1}{2};$ $b.\ \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$

13. Evaluate:

$a.\ 5\cdot(-3)^3+7$ $b.\ -2\cdot(-1.1)^2+15$ $c.\ 10-7\cdot(-2)^7$

$d.\ -20-10\cdot(-0.1)^2;$ $e.\ 7\cdot(-1)^3-4\cdot(-1)^2-8;$ $f.\ -10\cdot(-0.3)^2-5\cdot(-0.3)+1;$

14. Using the distributive property, express numerator as a product and simplify fractions:

Example:

$$\frac{33\cdot 6+33\cdot 7}{132\cdot 26}=\frac{33\cdot(6+7)}{132\cdot 26}=\frac{33\cdot 13}{33\cdot 4\cdot 2\cdot 13}=\frac{1}{8}$$

$a.\ \frac{14\cdot 5+14\cdot 7}{84\cdot 3};$ $b.\ \frac{18\cdot 9-18\cdot 4}{45\cdot 20};$

$c.\ \frac{16\cdot 13+16\cdot 12}{80\cdot 25};$ $d.\ \frac{27\cdot 15-27\cdot 7}{108\cdot 16};$

15. Ethan had $80 more than Emma. After Ethan spent half of his money, he had $10 less than Emma had. How much money did Emma and Ethan have altogether at the beginning?

16. Two cyclists started at the same time from two towns that are 232 km apart and rode toward each other. They met 8 hours later. If the first cyclist traveled at 15 km/h, find the speed of the second cyclist.

17. The numbers 100 and 90 were each divided by the same number. The remainder was 4 in the first case and 18 in the second. What number were they divided by?

18. Biologists were conducting a fish population study in a lake. They caught 80 fish, tagged them, and released them back into the water. The next day, they caught 80 fish again, and 5 of them were tagged. Assuming the tagged fish are evenly mixed with the rest of the fish, estimate the total number of fish in the lake.

19. Find x, if it's known, that the fractions are equal.

$a.\ \frac{x}{25}\ \ and\ \ \frac{12}{15};$ $b.\ \frac{9}{24}\ \ and\ \ \frac{x}{32};$ $c.\ \frac{18}{x}\ \ and\ \ \frac{8}{16};$ $d.\ \frac{14}{21}\ \ and\ \ \frac{22}{x}$