

Math 5b. Class work 2.



Negative Exponent.

If a number a in a power n is divided by the same number in a power m , the fraction can be written:

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}}{\underbrace{a \cdot a \cdot \dots \cdot a}_{m \text{ times}}} = \left(\underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}} \right) : \left(\underbrace{a \cdot a \cdot \dots \cdot a}_{m \text{ times}} \right) = a^n : a^m = a^{n-m}$$

General way to write this property can be a little confusing, so let's first work with natural numbers:

$$\frac{3^8}{3^5} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{3 \cdot 3 \cdot 3 \cdot \cancel{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}}{\cancel{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}} = 3^3 = 3^{8-5} = 3^{8+(-5)}$$

Can we now say, using the first property, that

$$3^{8+(-5)} = 3^8 \cdot 3^{(-5)}$$

Actually yes, it's very convenient, and also, we can see the division (which is the multiplication by the reciprocal of a number) by the same number several times as a power with negative exponent.

If a number is divided by itself, the result is one,

$$3:3 = \frac{3}{3} = 3 \cdot \frac{1}{3} = 3^1 \cdot 3^{-1} = 3^{1+(-1)} = 3^{1-1} = 3^0 = 1$$

And it's completely in agreement with the definition of 0 exponent.

So, we can say that

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a};$$

by definition.

$$5^{-7} = 5^{(-1) \cdot 7} = (5^{-1})^7 = \left(\frac{1}{5}\right)^7 = \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5} = \frac{1}{5^7}$$

And, of course, general way to write it is

$$a^{-m} = \frac{1}{a^m}; \quad a^{-m} \left(\frac{1}{a^m} \right) \quad \text{and} \quad a^m \quad \text{are reciprocal of each other}$$

Don't forget, a , if raised to a negative power, can't be zero, we can't divide by 0!

Let's see how our decimal system of writing numbers works when we use the concept of exponent not only for the whole part of the number, but also for its fractional part:

$$3456 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 = 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6$$

The value of a place of a digit is defined by a power of 10 multiplied by the digit. Very large numbers can be written using this system, as well as very small numbers.

$$0.3 = \frac{1}{10} \cdot 3 = \frac{1}{10^1} \cdot 3 = 10^{-1} \cdot 3$$

$$524.345 = 10^2 \cdot 5 + 10^1 \cdot 2 + 10^0 \cdot 4 + 10^{-1} \cdot 3 + 10^{-2} \cdot 4 + 10^{-3} \cdot 5$$

Scientists study objects that are very large and very small, from galaxies to viruses. They need to be able to write numbers, describing the object of interest, for example the distance between two galaxies or the diameter of a virus.

One of the most important numbers in the universe is the speed of light.

299 792 458 m / s. It's very convenient to represent it as a decimal between 1 and 10 and multiplied by a power of 10.

$$299\,792\,458 \text{ m per s} = 2.99792458 \cdot 10^8 \text{ m p s.}$$

Since 10^8 is a vary large number, it's safe to say, that the speed of light is very close to 300000000 m p s.

$$299\,792\,458 \text{ m per s} \approx 300000000 \text{ m per s} = 3 \cdot 10^8$$

Such representation is called scientific notation.

Let's convert the value to kilometers per hour. Each kilometer is 1000 meters, so we need to divide it by 1000:

$$3 \cdot \frac{10^8}{10^3} \text{ km p s} = 3 \cdot 10^{8-5} = 3 \cdot 10^5 \text{ km per s}$$

There are 3600 seconds in one hour, or $3.6 \cdot 10^3$ seconds. To find out the speed of light in km per hour we now need to multiply the speed in seconds by $3.6 \cdot 10^3$

$$\begin{aligned} 3 \cdot \frac{10^8}{10^3} \text{ km p s} &= 3 \cdot 10^{8-5} = 3 \cdot 10^5 \text{ km per s} = 3 \cdot 10^5 \cdot 3.6 \cdot 10^3 \text{ km p h} = 10.8 \cdot 10^8 \text{ km p h} \\ &= 1.08 \cdot 10 \cdot 10^8 = 1.08 \cdot 10^9 \text{ km p h} \end{aligned}$$



The Milky Way galaxy has a diameter of 105,700 light years, so the light will travel from one end to the other through its center in 105700 years.

How far is it from one side of the Milky Way to the other in kilometers?

$$1.08 \cdot 10^9 km \cdot 105700 \text{ years}$$

How many hours in a year? $24 \cdot 365.25 = 8766 \approx 8.8 \cdot 10^3 \text{ hours}$

$$1.08 \cdot 10^9 km \cdot 105700 \text{ years} \approx 1.08 \cdot 10^9 km \cdot 8.8 \cdot 10^3 \cdot 1.06 \cdot 10^5 \approx 10.07 \cdot 10^{17} km \\ \approx 10^{18} km.$$

This way to write numbers is called scientific notation, it's used a lot in science for describing various objects, big and small. The number should be represented as a product of one-digit whole-number part with decimal part rounded to two decimal places and a power of 10. For example, the distance from Earth to the Moon is about 238,855 miles (384,400 kilometers).

$$238855 = 2.38855 \cdot 10^5 \approx 2.39 \cdot 10^5 \text{ miles}$$

Let's take a look at some very small objects, such as bacteria and viruses.

$$1cm = 0.01m = \frac{1}{100}m = \frac{1}{10^2}m = 10^{-2}m$$

$$1mm = 0.001m = \frac{1}{1000}m = \frac{1}{10^3}m = 10^{-3}m$$

$$1\mu m = 10^{-6}m, \quad 1nm = 10^{-9}m$$

Bacteria are between $0.5 - 1.5 \mu m$

$$0.5\mu m = 0.5 \cdot 10^{-6}m = 5 \cdot 10^{-7}m$$

Viruses can be as small as $20 nm$ ($2 \cdot 10^{-8}m$) and as large (in the virus world, of course) as $300 nm$ ($3 \cdot 10^{-7}m$)

Problem solving examples:

Example 1:

Prove that

$8^5 + 2^{11}$ is divisible by 17

$$8^5 = (2 \cdot 2 \cdot 2)^5 = (2^3)^5 = 2^{15}$$

$$2^{15} = 2^{11+4} = 2^{11} \cdot 2^4$$

Therefore,

$$8^5 + 2^{11} = 2^{11} \cdot 2^4 + 2^{11}$$

2^{11} is a common factor and can be factor out:

BIG AND SMALL OBJECTS

Examples and their sizes in scientific notation

VERY BIG!

Diameter of the observable universe

$8.8 \times 10^{26} \text{ m}$

Diameter of the Sun

$1.39 \times 10^9 \text{ m}$

Diameter of the Earth

$1.27 \times 10^7 \text{ m}$

Height of Mount Everest

$8.85 \times 10^3 \text{ m}$

Length of a blue whale

$3.0 \times 10^1 \text{ m}$

Height of a person

$1.7 \times 10^0 \text{ m}$

VERY SMALL!

Diameter of a red blood cell

$7.0 \times 10^{-6} \text{ m}$

Length of a bacterium

$2.0 \times 10^{-6} \text{ m}$

Diameter of a virus

$1.0 \times 10^{-7} \text{ m}$

Width of a DNA molecule

$2.0 \times 10^{-9} \text{ m}$

Size of a water molecule

$2.7 \times 10^{-10} \text{ m}$

Size of an atom

$1.0 \times 10^{-10} \text{ m}$

Remember: In scientific notation, the number is written as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

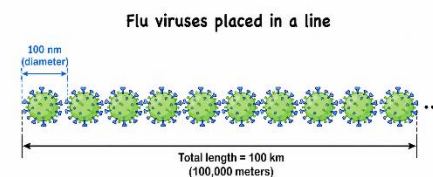
$$8^5 + 2^{11} = 2^{11} \cdot 2^4 + 2^{11} = 2^{11} \cdot (2^4 + 1) = 2^{11} \cdot (16 + 7) = 2^{11} \cdot 17$$

So, 17 is a factor of the value of the expression $8^5 + 2^{11}$, and this value is divisible by 17.

Example 2:

A flu virus is about 100 nm in diameter. During a typical flu infection, an infected person is estimated to produce about 10^{12} virus particles. If all of these virus particles were arranged in a straight line, touching one another, how long would the line be?

Express your answer in kilometers.



$$100 \text{ nm} = 100 \cdot 10^{-9} \text{ m} = 10^2 \cdot 10^{-9} \text{ m} = 10^{2-9} \text{ m} = 10^{-7} \text{ m}$$

$$10^{-7} \text{ m} \cdot 10^{12} = 10^{-7+12} \text{ m} = 10^5 \text{ m} = \frac{10^5}{10^3} \text{ km} = 10^2 \text{ km} = 100 \text{ km (about 62 miles)}$$

Example 3:

How many drops of water are in a swimming pool measuring $10 \text{ ft} \times 16 \text{ ft} \times 5 \text{ ft}$ if the volume of one drop of water is 0.05 cm^3 ? (1 foot equal to 30.48 cm)

How many cubic centimeters can fit into this pool?

$$10 \text{ feet} = 10 \cdot 30.48 \text{ cm} = 304.8 \text{ cm} = 3.048 \cdot 10^2 \text{ cm} \approx 3.05 \cdot 10^2 \text{ cm}$$

$$16 \text{ feet} = 16 \cdot 30.48 \text{ cm} = 487.68 \text{ cm} = 4.8768 \cdot 10^2 \text{ cm} \approx 4.88 \cdot 10^2 \text{ cm}$$

$$5 \text{ feet} = 5 \cdot 30.48 \text{ cm} = 152.4 \text{ cm} = 1.524 \cdot 10^2 \text{ cm} \approx 1.52 \cdot 10^2 \text{ cm}$$

$$10 \times 16 \times 5 \text{ cubic feet} = 3.05 \cdot 10^2 \cdot 4.88 \cdot 10^2 \cdot 1.52 \cdot 10^2 \text{ cm}^3 = 22.62368 \cdot 10^{2+2+2} \text{ cm}^3 \\ \approx 22.6 \cdot 10^6 \text{ cm}^3 = 2.26 \cdot 10^7 \text{ cm}^3$$

Now the only thing still unknown is how many drops fit into the volume of the pool.

$$\frac{2.26 \cdot 10^7 \text{ cm}^3}{0.05 \text{ cm}^3} = \frac{2.26 \cdot 10^7 \text{ cm}^3}{5 \cdot 10^{-2} \text{ cm}^3} = \frac{2.26}{5} \cdot \frac{10^7}{10^{-2}} = 0.452 \cdot 10^{7+2} = 4.52 \cdot 10^{-1} \cdot 10^9 \\ = 4.52 \cdot 10^{9-1} = 4.52 \cdot 10^8$$

Exercises:

1. Write a number in a power of (-1) :

Example:

$$5^{-1} = \frac{1}{5}; \quad \left(\frac{1}{7}\right)^{-1} = \frac{1}{\frac{1}{7}} = 1 \cdot 7 = 7; \quad \left(\frac{3}{4}\right)^{-1} = \frac{1}{\frac{3}{4}} = 1 \cdot \frac{4}{3} = \frac{4}{3};$$

$$2; \quad \frac{1}{2}; \quad 10; \quad \frac{1}{10}; \quad \frac{2}{3}; \quad \frac{3}{4};$$

2. Write as a power:

Example:

$$\frac{1}{2} = 2^{-1}; \quad \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$$

$$\frac{1}{3}; \quad \frac{1}{25}; \quad \frac{1}{27}; \quad \frac{1}{125}; \quad \frac{1}{81};$$

3. Express the following numbers as power:

$$128; \quad -64; \quad 0.0016; \quad -0.0016; \quad 0.0009; \quad -0.0009$$

4. Expand the numbers with power of 10:

Example:

$$\begin{aligned} 423.75 &= 100 \cdot 4 + 10 \cdot 2 + 1 \cdot 3 + \frac{1}{10} \cdot 7 + \frac{1}{100} \cdot 5 \\ &= 10^2 \cdot 4 + 10^1 \cdot 2 + 10^0 \cdot 3 + 10^{-1} \cdot 7 + 10^{-2} \cdot 5 \end{aligned}$$

$$12.32; \quad 32.1; \quad 456.789; \quad 0.321; \quad 562;$$

5. Write in scientific notation:

Example:

$$2347894532 \approx 2.35 \cdot 10^9; \quad 0.000675438 = 6.75438 \cdot 10^{-4} \approx 6.75 \cdot 10^{-4}$$

$$213451213; \quad 0.002315; \quad 123456789; \quad 0.1765;$$

6. Write the following numbers as the power of base 10:

$$10, 100, 1000, 10000, 100000, 1000000$$

$$0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001$$

7. Evaluate:

$$\begin{array}{cccccc} 2^5; & (-2)^5; & -2^5; & 3^4; & (-3)^4; & -3^4 \\ 0.2^6; & (-0.2)^6; & -0.2^6; & 0.05^2; & (-0.05)^2; & -0.05^2 \end{array}$$

$$10^7; \quad (-10)^7; \quad -10^7; \quad 0.1^3; \quad (-0.1)^3; \quad -0.1^3$$

8. A raindrop has a volume of about 0.05 cm^3 . How many raindrops are needed to fill a rain barrel that holds 200 liters? 1 liter equal to a 1000 cm^3 .
9. A drop of water has a volume of 0.05 cm^3 and contains about $2 \cdot 10^{21}$ water molecules. About how many water molecules are in a 2-liter bottle of water?
10. Prove that
- | | |
|-------------------------------------|-----------------------------------|
| $8^5 + 2^{11}$ is divisible by 17, | $9^7 - 3^{10}$ is divisible by 20 |
| $16^4 + 2^{12}$ is divisible by 17, | $4^3 - 2^4$ is divisible by 5 |

11. Reduce the fractions

a. $\frac{49^4 \cdot 7^5}{7^{12}};$

b. $\frac{3^{10} \cdot 27}{81^3};$

c. $\frac{125^3 \cdot 5^7}{5^{18}};$

d. $\frac{2^{15}}{16^3};$

e. $\frac{3^4 \cdot 9^5}{27^3};$

f. $\frac{25^4 \cdot 5^3}{5^{10}};$



Review Problems.

12. A can contained twice as much milk as a jar. After 2 liters were poured out of the jar and 3 liters were poured out of the can, the amount of milk remaining in the jar was 4.5 times smaller than the amount remaining in the can.
13. To enter the castle of Arithmos, in the Kingdom of Mathematica, you must crack the code: write the first 10 prime numbers in increasing order, using each number exactly once. This forms one long multi-digit number. Without changing the order of the remaining digits, cross out half of the digits so that the digits left form:
- the smallest possible number;
 - the largest possible number.

What are these two numbers?

14. Choose three values of n so that the product is divisible by the given number. Explain which values of n satisfy each condition.:

- a. $3n$ is divisible by 5.
- b. $12n$ is divisible by 4
- c. $5n$ is divisible by 15
- d. $6n$ is divisible by 9.

15. Evaluate (the answer is 0.8 (or $\frac{4}{5}$):

$$\frac{1\frac{1}{2} \cdot 2\frac{2}{3} \cdot 0.36}{0.6 \cdot 2\frac{1}{4} \cdot 1\frac{1}{3}};$$

16. Which digits should replace the letters A and B so that the following equality is true:

$$AB \cdot A \cdot B = BBB$$

(The same letters represent the same digits, and different letters represent different digits.)

17. Four people can paint a school auditorium in 10 days. How many days will it take 5 workers to paint the same auditorium if they work at the same rate?

18. Find 15% of the following numbers 125, 320, 24, 0.024

19. There were 24 sunny days in June. What percentage of the days in June were sunny?

20. Evaluate:

$$a. 1\frac{1}{12} \cdot 1\frac{1}{13} \cdot 1\frac{1}{14} \cdot 1\frac{1}{15} \cdot 1\frac{1}{16} \cdot 1\frac{1}{17}; \quad b. 4\frac{3}{7} \cdot 8\frac{4}{9} - 4\frac{3}{7} \cdot 6\frac{4}{9};$$

21. Which expression has the smaller value? Can you determine the answer without performing the calculations? Then evaluate all expressions to check your prediction..

$$a. 1 - \frac{1}{100}; \quad b. 1 - \left(\frac{1}{100}\right)^2; \quad c. \left(1 - \frac{1}{100}\right)^2$$