

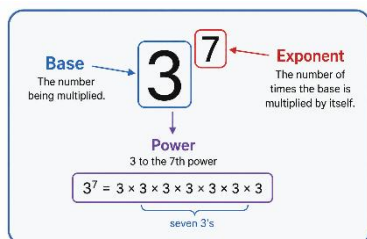
Math 5b. Class work 2.



Exponent.

Exponentiation is a mathematical operation, of multiplication of the same number several times, written as a^n , involving two numbers, the base a and the exponent n .

$$a^n = \underbrace{a \cdot a \cdot a \dots a}_{n \text{ times}}$$



In that case, a^n is called the n -th power of a , or a raised to the power n .

The exponent indicates how many copies of the base are multiplied, in a very similar way as a product represent indicates how many copies of a number are added together.

For example, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$. The base 3 appears 5 times in the repeated multiplication, because the exponent is 5. Here, 3 is the *base*, 5 is the *exponent*, and 243 is the *power* or, more specifically, *the fifth power of 3*, *3 raised to the fifth power*, or *3 to the power of 5*.

Another example, $7^4 = 7 \cdot 7 \cdot 7 \cdot 7 = 2401$. In this case, 7 is a base and 4 is the exponent.

Properties of natural exponent.

- To multiply two powers with the same base we can write:

$$3^{10} \cdot 3^{15} = \underbrace{(3 \cdot 3 \dots 3)}_{10 \text{ times}} \cdot \underbrace{(3 \cdot 3 \dots 3)}_{15 \text{ times}} = \underbrace{3 \cdot 3 \cdot 3 \dots 3}_{10+15 \text{ times}} = 3^{10+15} = 3^{25}$$

The total number of 3s multiplied is exactly $10 + 15 = 25$

In general form the same expression can be written as following:

$$a^m \cdot a^n = \underbrace{a \cdot a \dots a}_m \cdot \underbrace{a \cdot a \dots a}_n = \underbrace{a \cdot a \cdot a \dots a}_{m+n \text{ times}} = a^{m+n}$$

(for now, let assume that n and m are both natural numbers)

- To raise an exponent to another exponent we can write;

$$(3^{10})^{15} = \underbrace{3^{10} \cdot 3^{10} \cdot \dots \cdot 3^{10}}_{15 \text{ times}} = \underbrace{\underbrace{3 \cdot 3 \cdot \dots \cdot 3}_{10 \text{ times}} \cdot \dots \cdot \underbrace{3 \cdot 3 \cdot \dots \cdot 3}_{10 \text{ times}}}_{15 \text{ times}} = 3^{10 \cdot 15} = 3^{150}$$

In this case, total number of 3s multiplied between them is $10 \cdot 15 = 150$

General expression will look like this:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \dots \cdot a^m}_n = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_m \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_m}_{m \text{ times}} = a^{m \cdot n}$$

- By multiplying an exponent by one more base we are going to get the expression:

$$3^{10} \cdot 3 = \underbrace{3 \cdot 3 \cdot 3 \dots 3}_{10 \text{ times}} \cdot 3 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \dots 3}_{10+1 \text{ times}} = 3^{10+1} = 3^{10} \cdot 3^1$$

As you can see, number can be considered as a number in the first power, and this assumption is very well aligned with first two properties of the operation of raising to a power:

$$3^{10} \cdot 3 = 3^{10+1} = 3^{10} \cdot 3^1$$

and

$$(3^{10})^1 = 3^{10 \cdot 1} = 3^{10}$$

General way to write this is

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \dots a}_{n \text{ times}} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \dots a}_{n+1 \text{ times}} = a^{n+1} = a^n \cdot a^1$$

Any number in the first power is equal to itself.

In order to have the set of power properties consistent, $a^1 = a$ for any number a .

Next step is a little more sophisticated.

- Let's multiply an exponent by 1. We always can multiply by 1, by identity property of multiplication, nothing will change. But now we know the first property of power and can do this:

$$3^{10} \cdot 1 = 3^{10} = 3^{10+0} = 3^{10} \cdot 3^0$$

Or, in general form:

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0$$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number a , but 0 (for any $a \neq 0$)

- For two numbers, raised to the same power we can write

$$(2 \cdot 3)^{10} = \underbrace{(2 \cdot 3) \cdot \dots \cdot (2 \cdot 3)}_{10 \text{ times}} = \underbrace{2 \cdot \dots \cdot 2}_{10 \text{ times}} \cdot \underbrace{3 \cdot \dots \cdot 3}_{10 \text{ times}} = 2^{10} \cdot 3^{10}$$

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \dots \cdot (a \cdot b)}_{n \text{ times}} = \underbrace{a \cdot \dots \cdot a}_{n \text{ times}} \cdot \underbrace{b \cdot \dots \cdot b}_{n \text{ times}} = a^n \cdot b^n$$

A positive number raised into any power will result a positive number; the product of any positive numbers, regardless of how many of them are multiplied, will be positive.

A negative number, raised in a power, represented by an even number is positive:

$$\begin{aligned} (-5)^6 &= (-5) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) = \\ &= \left(\underbrace{(-5) \cdot (-5)}_{25} \right) \cdot \left(\underbrace{(-5) \cdot (-5)}_{25} \right) \cdot \left(\underbrace{(-5) \cdot (-5)}_{25} \right) = 15625 \end{aligned}$$

1	$a^n = \underbrace{a \cdot a \cdot a \dots a}_{n \text{ times}}$	Definition of a power. The base a is multiplied by itself n times.
2	$a^n \cdot a^m = a^{n+m}$	Product of powers with the same base. Add the exponents.
3	$(a^n)^m = a^{n \cdot m}$	Power of a power. Multiply the exponents.
4	$a^1 = a$, for any a	First power. Any number to the first power is itself.
5	$a^0 = 1$, for any $a \neq 0$	Zero power. Any nonzero number to the zero power is 1.
6	$(a \cdot b)^n = a^n \cdot b^n$	Power of a product. Raise each factor to the power.

because a few pairs of negative numbers are multiplied, each pair produce a positive number, and a final result is positive.

A negative number, raised in a power, represented by an odd number is negative:

$$\begin{aligned} (-5)^7 &= (-5) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) = \\ &= \left(\underbrace{(-5) \cdot (-5)}_{25} \right) \cdot \left(\underbrace{(-5) \cdot (-5)}_{25} \right) \cdot \left(\underbrace{(-5) \cdot (-5)}_{25} \right) \cdot (-5) = 15625 \cdot (-5) \end{aligned}$$

Odd number is always an even number plus 1 and a resulting product is a product of a positive and a negative number, so, it's negative.

When you write something as a power, be careful with notation. Look at the following examples:

$$\begin{aligned} (-5)^4 &= (-5) \cdot (-5) \cdot (-5) \cdot (-5) = 625, \text{ but } -5^4 = -5 \cdot 5 \cdot 5 \cdot 5 = -625 \\ 3 \cdot 4^2 &= 3 \cdot 4 \cdot 4 = 48 \quad \text{but} \quad (3 \cdot 4)^2 = (3 \cdot 4) \cdot (3 \cdot 4) = 12 \cdot 12 = 144 \end{aligned}$$

Exponents can help write numbers in their expanded form. In our base-10 place value system, the position of the digit indicates which power of 10 should be multiplied by the digit. For example:

$$345 = 300 + 40 + 5 = 100 \cdot 3 + 10 \cdot 4 + 1 \cdot 5 = 10^2 \cdot 3 + 10^1 \cdot 4 + 10^0 \cdot 5$$

Exponent is very interesting mathematical operation. There is the story of the invention of the game of chess. The king ordered a new game because he was bored by the old games, was so happy about the new chess game that he said to the inventor: *"Name your reward and you will get it!"* The inventor asked for a simple reward. *"I would like to have one grain of rice on the first chess square, two on the second, four on the third and so on, doubling the amount of rice every square."* The legend says that the King was surprised he didn't ask for gold but was quite content that the inventor asked for so little. But when the court scholars told him there wasn't enough rice in the whole world to fill the chess board, he had to admit his loss:

$$1 + 2 + 2^2 + 2^4 + \dots + 2^{63} = 18,446,744,073,709,551,615$$

The weight of the rice grain is about 0.03g. so:

$$1.8 \cdot 10^{19} \cdot 0.03 = 5.4 \cdot 10^{17} \text{ g. or about } 5.4 \cdot 10^{14} \text{ kg or } 10^{15} \text{ lb.}$$

That's about 3,000 Olympic swimming pools full of rice.

Another example:

If I put 1000 dollars into the bank account and will get 5% annually (I will not take out anything).
In 1 year, my deposit will become

$$1.05 \cdot 1000 = 1050$$

Next year:

$$1.05 \cdot 1.05 \cdot 1000 = 1.05^2 \cdot 1000 = 1102.5$$

In 100 years:

$$1.05^{100} \cdot 1000 \approx 131.5 \cdot 1000 = 131500$$

Let's see a few examples of solving problems, using the properties of exponent.

Example 1.

Compare 26^9 and 82^7 .

At first glance, the problem seems difficult, calculation of 26 in the 9th power is not a simple task (supposed we don't have a calculator). But we know a lot about exponent. First think which can be noted, that

$$26^9 < 27^9$$

You can see it right away, using the definition of exponent: 26 multiplied 9 times will be smaller than 27 multiplied also 9 times.

$$27^9 = (3^3)^9 = 3^{3 \cdot 9} = 3^{27}$$

Therefore,

$$26^9 < 3^{27}$$

On the other hand,

$$82^7 > 81^7$$

$$81^7 = (3^4)^7 = 3^{4 \cdot 7} = 3^{28}$$

$$82^7 > 3^{28}$$

And, the last step:

$$26^9 < 3^{27} < 3^{28} < 82^7$$

Example 2.

What would be the last digit of numbers 5^{27} , 2^{75} ?

It's obvious, that if 5 is multiply any number of times, the last digit will always be 5. What about 2?

Take a look on a table below:

2^1	2^2	2^3	2^4	2^5	2^6	2^7
2	4	8	16	32	64	128

Do you see the pattern? Until the fifth power all the digits are different. The fifth power of 2 is 16 (4th power) multiplied by 2, and the last digit of this 2-digit number is 2. Obviously, 6th power will end with 4, the next one with 8 and so on. Patter (2, 4, 8, 6) will repeat itself. Let see, how many times this pattern can fit into 75.

$$75:4 = 18R3$$

Pattern will repeat itself exactly 18 times and will stop at the third element, digit 8, the last digit of 2^{75} is 8.

 COMMON MISTAKES	
Be careful! Here are some common mistakes when working with exponents.	
 Correct	 Incorrect
$2^3 = 2 \times 2 \times 2 = 8$	$2^3 = 2 \times 3$
$(-4)^2 = 16$	$-4^2 = 16$
$5^1 = 5$	$5^1 = 1$
$7^0 = 1$	$7^0 = 0$
 Remember: The exponent tells how many times the base is used as a factor.	

Exercises:

1. Write as a product:

a. $3 + 3 + 3 + 3$;

b. $7 + 7 + 7 + 7 + 7 + 7$;

c. $4 + 4 + 4 + 4 + 4 + 4$;

2. Write as a power:

Example: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$

a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$;

b. $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$;

c. $4 \cdot 4 \cdot 4 \cdot 4$;

3. Express each product as a power:

a. $5 \cdot 5$;

b. $10 \cdot 10 \cdot 10 \cdot 10$;

c. $a \cdot a \cdot a \cdot a$;

d. $x \cdot x \cdot x \cdot x \cdot x$

4. Express each power as a product:

Example: $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$

a. 5^7 ;

b. 6^3 ;

c. 9^2 ;

d. m^7 ;

5. Simplify the expressions, write as a power:

Example: $13 \cdot 13 \cdot 5 \cdot 5 \cdot 5 = 13^2 \cdot 5^3$

a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$;

b. $12 \cdot 7 \cdot 7 \cdot 7 \cdot 7$;

c. $(3 \cdot 5) \cdot (3 \cdot 5) \cdot (3 \cdot 5) \cdot (3 \cdot 5)$;

d. $(-2) \cdot (-2) \cdot 3 \cdot 3 \cdot 3$;

e. $-5 \cdot 5 \cdot 9 \cdot 9 \cdot 9$;

f. $(2 \cdot 7) \cdot (2 \cdot 7) \cdot (3 \cdot 9) \cdot (3 \cdot 9)$;

6. Compare:

a. 5^3 and $5 \cdot 3$;

b. 12^2 and $12 \cdot 2$;

c. 2^5 and 5^2 ;

d. 3^{10} and 3^{11} ;

e. 5^{21} and 6^{21} ;

f. 2^4 and 4^2 ;

7. What should be the exponent for the equation to hold?

a. $8^* = 512$; b. $2^* = 64$; c. $3^* = 81$; d. $7^* = 343$

8. What digits should be put instead of * to get true equality? How many solutions does each problem have?

a. $(2 *)^2 = ** 1$; b. $(3 *)^2 = *** 6$

c. $(7 *)^2 = *** 5$ d. $(2 *)^2 = ** 9$, e. $(3 *)^2 = ** 1$

9. In a magical lake, the number of water lilies doubles every night. On March 1st, the magician planted the first lily, and in 90 nights, the entire lake was covered with lilies. On which day was only half of the lake covered?



10. Evaluate:

$$(-3)^2; \quad -3^2; \quad (-3)^3; \quad 2^7; \quad (-2)^7; \quad -2^7; \quad (2 \cdot 3)^3; \quad 2 \cdot 3^3; \quad \left(\frac{1}{3}\right)^2; \quad \frac{1}{3^2};$$

11. Represent numbers as a power of 10:

Example: $1000^3 = (10^3)^3 = 10^{3 \cdot 3} = 10^9$

100^2 ; 100^3 ; 100^4 ; 100^5 ; 100^6 ;

12. Write the numbers in their extended form;

231, 305, 2001, 2931, 3400.

13. Write the number which extended form is written below;

Example: $2 \cdot 10^3 + 7 \cdot 10^2 + 2 \cdot 10^1 + 6 \cdot 10^0 = 2726$;

a. $3 \cdot 10^3 + 5 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0$;

b. $6 \cdot 10^3 + 1 \cdot 10^2 + 0 \cdot 10^1 + 3 \cdot 10^0$;

c. $8 \cdot 10^3 + 5 \cdot 10^1 + 3 \cdot 10^0$;

e. $3 \cdot 10^3 + 2 \cdot 10^2 + 5 \cdot 10^1 + 4 \cdot 10^0$;

14. Compare:

a. 5^3 and $5 \cdot 3$;

b. 12^2 and $12 \cdot 2$;

c. 2^5 and 5^2 ;

d. 3^4 and 4^3 ;

e. 2^2 and $2 \cdot 2$;

f. 2^4 and 4^2 ;

15. Which number is greater?

a. 29^2 or 1000;

b. 43^2 or 1500;

c. 49^2 or 3000;

d. 67^2 or 3500

16. Write the following expressions in a shorter way replacing product with power:

Examples:

$$(-a) \cdot (-a) \cdot (-a) \cdot (-a) = (-a)^4, \quad 3m \cdot m \cdot m \cdot 2k \cdot k \cdot k \cdot k = 6m^3k^4$$

$$a. (-y) \cdot (-y) \cdot (-y) \cdot (-y);$$

$$b. (-5m)(-5m) \cdot 2n \cdot 2n \cdot 2n;$$

$$c. -y \cdot y \cdot y \cdot y;$$

$$d. -5m \cdot m \cdot 2n \cdot n \cdot n;$$

$$e. (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab);$$

$$f. p - q \cdot q \cdot q \cdot q \cdot q;$$

$$g. a \cdot b \cdot b \cdot b \cdot b \cdot b;$$

$$h. (p - q) \cdot (p - q) \cdot (p - q);$$

17. Simplify the expressions:

$$a. 2^4 + 2^4;$$

$$b. 2^m + 2^m;$$

$$c. 2^m \cdot 2^m;$$

$$d. 3^2 + 3^2 + 3^2;$$

$$e. 3^k + 3^k + 3^k;$$

$$f. 3^k \cdot 3^k \cdot 3^k;$$

18. What will be last digit of

$$a. 2^{22};$$

$$b. 3^{33};$$

$$c. 4^{44};$$

$$d. 5^{55};$$

$$e. 6^{66};$$

$$f. 7^{77};$$

19. Compare:

$$a. 127^{23} \text{ and } 513^{18}$$

$$b. 9997^{10} \text{ and } 100003^8$$

$$c. 5^{300} \text{ and } 3^{500}$$

20. Write the following numbers as the power of base 10:

$$10, 100, 1000, 10000, 100000, 1000000$$

Review Problems.

21. Evaluate the expressions and compare their values;

$$a. 5 \cdot (8 + 14) \text{ and } 5 \cdot 8 + 14$$

$$b. 5 \cdot (8 + 14) \text{ and } 5 \cdot 8 + 5 \cdot 14$$

$$c. 5 \cdot (6 + 4) \cdot 25 \text{ and } 5 \cdot 6 + 4 \cdot 25$$

$$d. 5 \cdot (20 - 6) + 40 \text{ and } 5 \cdot 20 - 6 + 40$$

22. The population of a town increased by 10% in the first year, and then by another 20% in the second year. By what overall percentage did the population increase over two years?

23. Evaluate (try to do it by the easiest possible way):

$$\frac{7777777 \cdot 7777777}{1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1} - \frac{5555555 \cdot 5555555}{1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1}$$

24. x is a natural number.

Among following statements 3 are true and 2 are false.

$2 \cdot x$ is greater than 70

x is less than 100

$3 \cdot x$ is greater than 25

x is not less than 10

x is greater than 5

What is x ?

25. Put parenthesis so, that equality is true.

a. $5 \cdot 38 - 70 : 8 - 6 = 60;$

b. $630 : 7 : 2 \cdot 9 \cdot 25 = 125;$

c. $30 - 49 : 42 : 6 \cdot 8 = 184;$

d. $180 : 300 - 30 \cdot 9 + 199 = 125;$

26. Find all possible divisors of number 56. Write them in increasing order.

27. Two cyclists are riding along the same road. At the moment, the distance between them is 54 km. One cyclist is traveling at 12 km/h, which is $\frac{3}{4}$ of the other cyclist's speed. What will the distance between them be after 2 hours? (*Consider all possible cases.*)

