Homework 2.

Electrostatic potential energy

While the electric force between two charges $\boldsymbol{Q}$ and $\boldsymbol{q}$ can be expressed as:

$$\left|\vec{F}\right| = k\frac{Q\cdot q}{r^2}\cdot \quad (1),$$

where r is the distance between the charges, the potential energy of the charges is

$$E_P = k\frac{Q\cdot q}{r}\cdot \quad (2).$$

Note that it is just r rather than r^2 in the denominator.
How we can obtain this formula?
Let us take two small charged objects – "point charges", with charges $\boldsymbol{Q}$ and $\boldsymbol{q}$, both being positive and separated by a distance $\boldsymbol{r_1}$ (Figure 1).

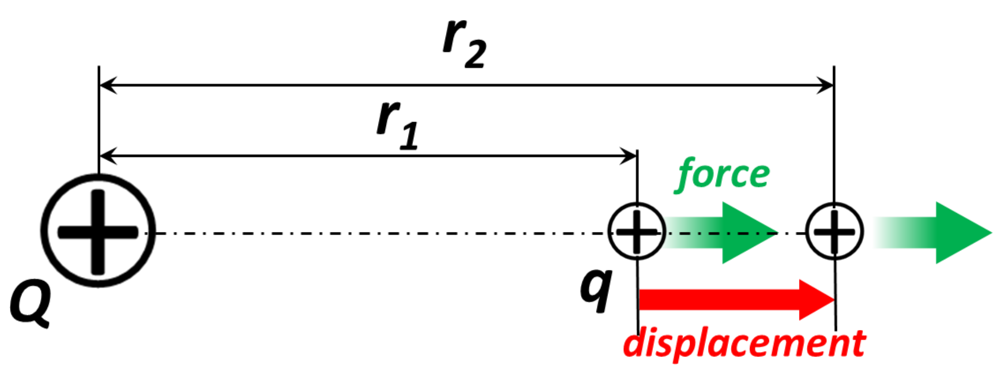


Figure 1.

Initially, the positions of both charges are fixed, the charges have some potential energy and their kinetic energy is zero. Then we release charge $\boldsymbol{q}$. This charge, being repelled by charge $\boldsymbol{Q}$, starts moving. The distance, separating the charges, increases from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ and kinetic energy of charge $\boldsymbol{q}$ increases. According to work-energy theorem, the change in kinetic energy is equal to the work, done by Coulomb force on charge $\boldsymbol{q}$. On the other hand, the total energy of charge $\boldsymbol{q}$ stays the same. (The position of charge Q stays fixed - we are "holding it" – so its kinetic energy is still zero). So, the work W, i.e. the increase of the kinetic energy ($\boldsymbol{E_{K2}}$-$\boldsymbol{E_{K1}}$) is equal to the decrease of the potential energy ($\boldsymbol{E_{P1}}$-$\boldsymbol{E_{P2}}$):

$$E_{P1} - E_{P2} = work = force \cdot displacement\ along\ the\ force$$

$$E_{P1} - E_{P2} = k\frac{Q\cdot q}{r^2}\cdot (r_2 - r_1) \quad (3)$$

But there is one difficulty: the Coulomb force depends on the distance ***r*** between the charges and this distance changes from r_1 to r_2 as charge ***q*** moves away ☹. Which number I should plug instead of ***r*** in the denominator of the expression for the force?
So, what I suggest:

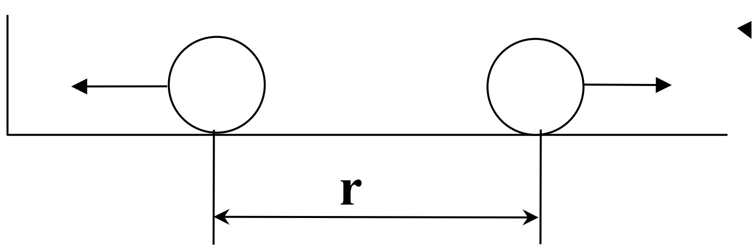
1. Let us assume that the displacement of the charge ***q*** is very small, so $\boldsymbol{r_1 \approx r_2}$
2. Let is replace r^2 in the denominator of expression (3) to r_1r_2. We have to use square of the ***average*** distance but r_1 is a bit less than the average distance and r_2 is a bit higher than the average distance, so his replacement seems to be OK:

$$\boldsymbol{E_{P1} - E_{P2} = k\frac{Q \cdot q}{r_1 \cdot r_2} \cdot (r_2 - r_1) = kQq \cdot \left(\frac{r_2}{r_1 \cdot r_2} - \frac{r_1}{r_1 \cdot r_2}\right)}$$

$$\boldsymbol{E_{P1} - E_{P2} = k\frac{Q \cdot q}{r_1} - k\frac{Q \cdot q}{r_2}} \quad (4)$$

So, finally we came to formula (2) for electrostatic potential energy.

If two charges are of different signs than the potential energy is *negative*. It just means that the closer two charges are, the less is their potential energy. The charges with the opposite signs "like" to be as close as possible to reduce their potential energy. So, negative electrostatic potential energy means attraction. For two charges of same sign, the potential energy is *positive* and increases with the distance between them. It means repulsion.
Here is an example:
*Problem: Imagine that we have two identical negatively charged balls (the mass is **M** charge is **q**) are separated by the distance r_{before}. We let the balls go and they start moving. What are the velocities **v** of the balls when the distance between them is r_{after}?*



Solution:
We can try to use Coulomb's law to calculate the force applied to each ball, find acceleration of each ball and, using kinematics formula calculate the time and final velocities. This is a long way, and, moreover, soon we will meet a serious difficulty – the interaction force and acceleration change with the distance.
There is another, much simpler solution which is based on the energy conservation law. As the balls move away from each other their potential energy decreases, but the kinetic energy of both ball increases. Total energy conserves so the increase of kinetic energy equals to decrease of potential energy. Since the balls are identical, each of them gets halve of the total kinetic energy (symmetry consideration):

$$k\frac{q\cdot q}{r_{after}}+2M\frac{v^2}{2}$$ - total energy when the distance is 20m

(!) ***we do not multiply the potential energy by 2 since this is "joint" energy of the system of 2 charges. However, we multiply the kinetic energy by 2 since M is the mass of one ball.***

$$k\frac{q\cdot q}{r_{before}}$$ - total energy in the beginning, when the distance is 10m

$$k\frac{q\cdot q}{r_{after}}+2M\frac{v^2}{2}=k\frac{q\cdot q}{r_{before}}$$ - energy conservation

In this equation we know everything except v – so we can easily calculate it.

Electric potential

The potential energy where of two charges separated by a distance r is

$$P=k\frac{q_1\cdot q_2}{r} \quad (1)$$

Let us keep one of the charges, say, q_1 fixed and change the charge q_2. Since there is a product of the charge magnitudes in the numerator of formula (1), the potential energy will increase or decrease proportionally to the charge magnitude of q_2. We can now calculate the potential energy *per unit charge*. For this we will divide the potential energy of the interacting charges q_1 and q_2 by the magnitude of q_2:

$$\frac{P}{q_2}=k\frac{q_1\cdot q_2}{r}\div q_2=k\frac{q_1}{r} \quad (2)$$

We can imagine that each point of space around the charge q_1 can be characterized by the potential energy of a positive unit charge in this point. The electrostatic potential energy of a positive unit charge in a certain point is called "***electric potential***" in this point. The electric potential is a scalar. The electric potential $\boldsymbol{\varphi}$ created by the point charge $\boldsymbol{Q}$ is:

$$\boldsymbol{\varphi=k\frac{Q}{r}} \quad (3)$$

There are 2 important issues. First: if the charge q is negative, the potential will be negative as well. Second: potential, created by a point charge is "spherically symmetrical". This means that only the distance from the charge to the point where we are measuring potential matters (rather than the exact position of the point).

The formula (3) means that a unit positive point charge placed at the distance r from the charge q will have potential energy $\boldsymbol{\varphi}$. If we will place an arbitrary charge Q at the distance r (instead of a unit charge) then the potential energy of the charge Q can be calculated as:

$$\boldsymbol{P = k\frac{Q}{r} \cdot q = \varphi \cdot q} \quad (4)$$

As we can see from the formula (3) the potential created by a point charge depends on the distance to the point charge. Difference of potentials taken in points A and B equals to the difference of potential energy of a unit positive charge in these points. Now let us look at Figure 1. Let us assume that the position of positive charge $\boldsymbol{Q}$ is fixed, and another small object having charge $\boldsymbol{q}$ is placed in a point separated from $\boldsymbol{Q}$ with a distance $\boldsymbol{R_1}$. Charge $\boldsymbol{q}$ is repelled by charge Q, and, being released, starts moving from charge $\boldsymbol{Q}$. When it reaches point B, separated from $\boldsymbol{Q}$ with a distance $\boldsymbol{R_2}$, it has nonzero kinetic energy, but its potential energy is less now. The gain in kinetic energy is equal to the loss of the potential one due to energy conservation.

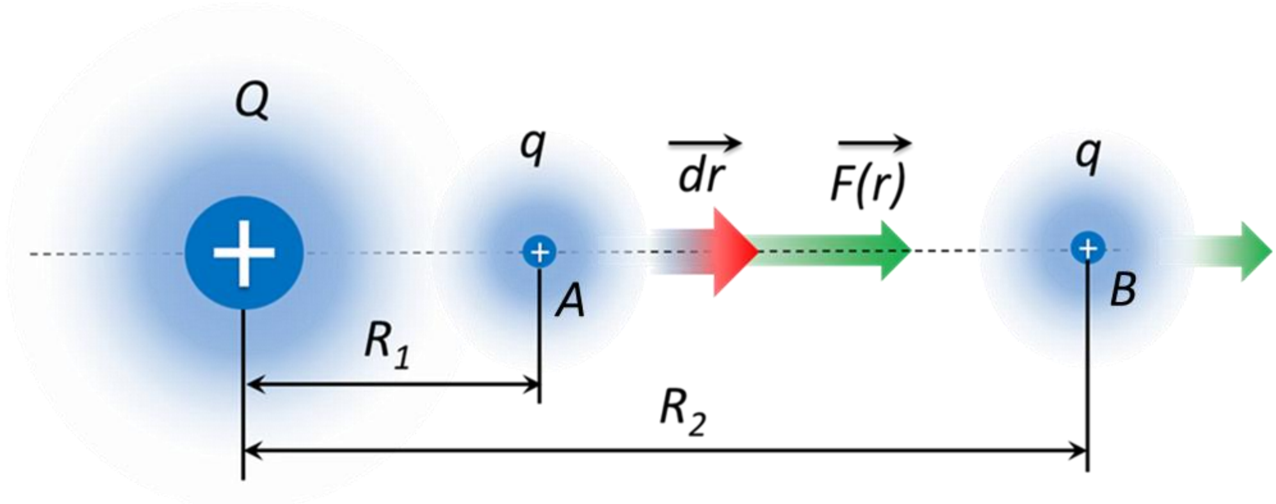


Figure 1. Work of the electric field. dr is a displacement in the direction AB, F(r) is the Coulomb force which depends on r.

But, the gain in kinetic energy is equal to the work $\boldsymbol{W_{AB}}$, done by electrostatic force on charge q. Typically, by "change" we mean "final value minus initial value" Change in potential energy then will be negative since its final valuer at $\boldsymbol{R_2}$ is less than its initial value at $\boldsymbol{R_1}$ (since $\boldsymbol{R_2}$ is larger than $\boldsymbol{R_1}$). Let us introduce a slightly different parameter: potential energy in the *initial* point minus potential energy in the *final* point. We have:

$$\boldsymbol{W_{AB} = P_A - P_B = q\varphi_A - q\varphi_B = q(\varphi_A - \varphi_B) = qU_{AB}} \qquad (5)$$

Here P_A, P_B –electrostatic potential energies in points A and B; $\boldsymbol{\varphi_A}$, $\boldsymbol{\varphi_B}$ – the electrostatic potentials, $U_{AB} = \boldsymbol{\varphi_A - \varphi_B}$ is the potential difference which is also called "***voltage drop** between points A and B"*, or just "***voltage** between points A and B"*.

We assumed that the charge was moved from point A to point B along a straight line. We may ask: "may be there is a an optimal path, so if we use this path the loss of potential energy when the charge moved from one point to the other will be minimal". This does not work for electrostatic field. **The potential energy (or just potential) difference between two points does not depend on the path we choose!**

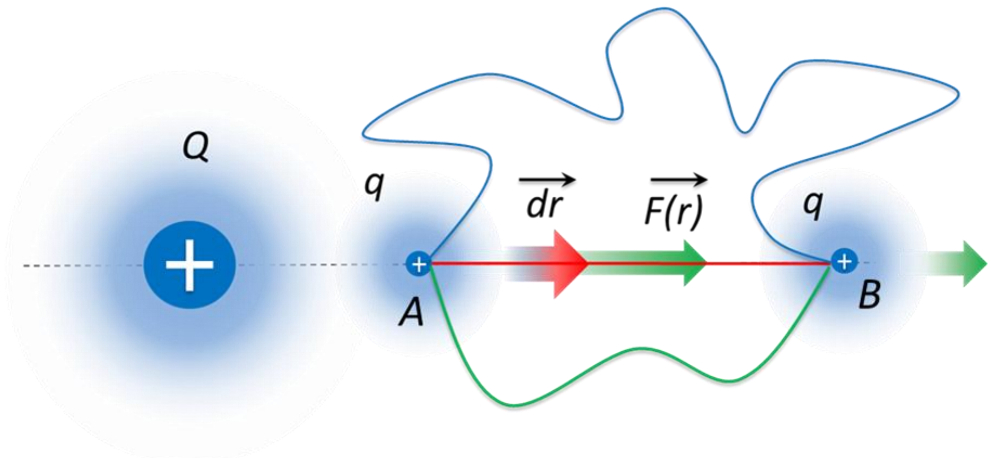
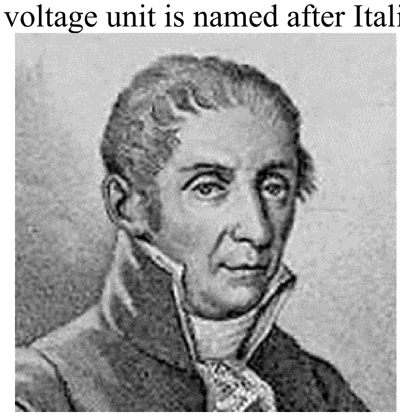


Figure 2. Potential energy difference between points A and B does not depend on the path geometry.

We can choose the green path (Figure 2) or the blue pass the potential energy difference will be the same. Let us assume for a moment that the change in potential energy depends on the path you choose and for the blue path it is, say, higher that for the green path. Then we could choose green path to go from A to B and blue pass to go from B to A and return to the same point A having higher potential energy than we initially had in this point. But this does not agree with the expression for electrostatic potential energy we obtained earlier. According to this expression if the distance is the same so is the potential energy. The electrostatic potential and voltage are measured in Volts.

1Volt=1Joule/1Coulomb. The voltage unit is named after Italian physicist Alessandro Volta:



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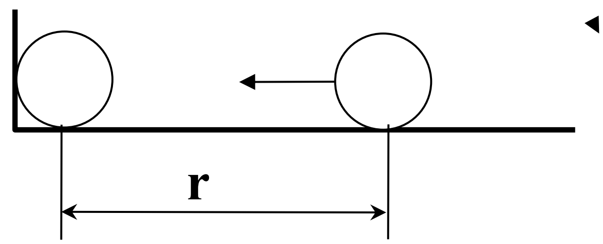
The homework problems are below:

1. A 2kg positively charged (the charge is 0.001C) ball is pushed toward identical ball with the same charge. The position of the second ball is fixed. When the distance between the balls is 10m, the speed of the first ball is 1m/s. Find the distance at which the moving ball stops?



2. 4 identical charges 10^{-3}C each are placed and held in the corners of square with the side of 1 meter. Find electrostatic potential energy of the system.

3. Two metal balls with masses 2kg and 1 kg, are charged. The charge of the first ball is –0.1C, the charge of the second is +0.3C. The balls are separated by a distance of 5m. After the balls are released, they start moving toward each other. Find the velocities of the balls when the distance between them is 2m. (Hint use both energy and momentum conservation laws)

4. An object with a charge of 0.01C being accelerated by electrostatic force moves from point A to point B and gains kinetic energy of 6J. Find the potential difference between points A and B.

5. There is a point charge of -1C (see picture below). The distance between the charge and the point A is 100m, the distance between the points A and B is also 100m. Find the potential difference between points A and B.