Homework 19

Rolling down incline plane

Let us consider an object which can roll downhill. The object rolls without sliding down an incline plane which forms angle $\boldsymbol{\alpha}$ with the horizon. We assume that the object has rotational symmetry, so as it rolls down, its center of mass is moving parallel to the slope. It may be a solid or hollow ball, a cylinder etc. The moment of inertia (rotational mass) of the object with respect to the axis passing through the center of mass is $\boldsymbol{I_C}$. Let us find acceleration of the center of mass of the object.

First we plot a force diagram. It is shown in Figure 1

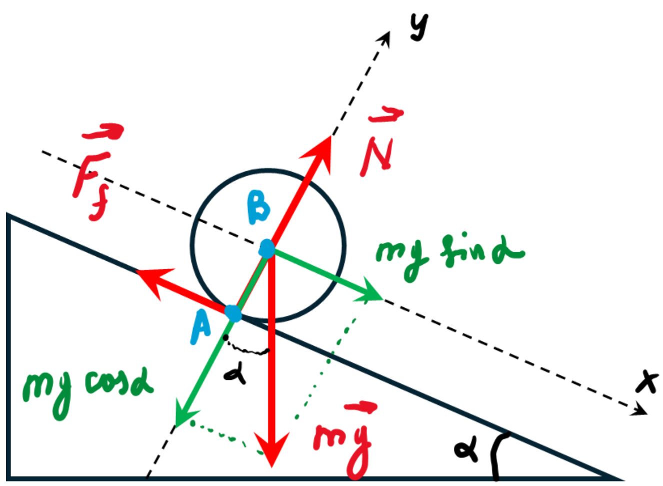

Figure 1, Force diagram of the rolling object.

Here $\boldsymbol{N}$ is the normal force and $\boldsymbol{F_f}$ is the friction force. Then we can write second Newton's law for the acceleration $\boldsymbol{a}$ directed along x axis:

$$ma = -F_f + mg \cdot sin\alpha \qquad (1)$$

We have 2 unknown variables – $\boldsymbol{F_f}$ and $\boldsymbol{a}$, so another equation is needed. This may be a second Newton's law for rotation with respect to axis B, which passes through the center of mass of the object:

$$I_C\beta = F_f \cdot r \qquad (2),$$

Here $\boldsymbol{r}$ is the radius of the rolling object, $\boldsymbol{\beta} = \frac{a}{r}$, is its angular acceleration. Expressing $\boldsymbol{F_f}$ from equation (2) and plugging it to equation (1) gives:

$$a = \frac{g \cdot sin\alpha}{1 + \frac{I_C}{mr^2}} \qquad (3)$$

We can introduce a new parameter $\rho = \sqrt{\frac{I_C}{m}}$, which is called radius of inertia. Then expression 1 can be rewritten as:

$$a = \frac{g \cdot sin\alpha}{1+\frac{\rho^2}{r^2}} \qquad (4)$$

We can also calculate the rolling friction force from (1) and (2):

$$F_f = \frac{mg \cdot sin\alpha}{1+\frac{r^2}{\rho^2}} \qquad (5)$$

Now, let us try to solve the same problem using a simpler way. Let us write the second Newton's law for rotational motion with respect to axis A. We remember that rolling can be thought of as rotation around instantaneous axis passing through A. In this case, the momentum of the friction force is zero and we need just one equation. But the problem is that rotational mass with respect to axis A, $\boldsymbol{I_A}$, is not equal to $\boldsymbol{I_C}$. However, we can easily calculate it using Huygens-Steiner theorem (parallel axis theorem):

"The moment of inertia I_A of a rigid body about any axis is equal to its moment of inertia about a parallel axis through its center of mass I_C plus the product of the mass m and the square of the distance h between the axes:

$$\boldsymbol{I_A = I_C + mh^2} \qquad \textbf{(6)"}$$

Then we have:

$$I_A\beta = mg \ \cdot sin\alpha \cdot r \qquad (7), \text{ so}$$

Using (6) we have:

$$a = \beta \cdot r = \frac{mg \cdot sin\alpha \cdot r^2}{I_C + mr^2} \qquad (8),$$

Then, dividing numerator and denominator of fraction (8) by mr^2 , we obtain same equation as (4).

As the object rolls without sliding, friction force $\boldsymbol{F_f}$ has to be less than μN, where $\boldsymbol{\mu}$ is the friction coefficient. If rolling required friction force exceeding μN , then rolling without sliding is impossible and the object will have bot sliding and rolling : the lower point of the object is not anymore at rest with respect to the plane.

Problem:

1. A solid ball with radius $\boldsymbol{R}$ and mass $\boldsymbol{m}$ rolls down an incline plane which forms angle $\boldsymbol{\alpha}$ with the horizon. The friction coefficient between the ball and the plane is $\boldsymbol{\mu}$. Find the critical incline angle at which the ball starts sliding.