Homework for February 1, 2025.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.
2. Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?
3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
    a. no letter will be put into the envelope with its correct address?
    b. only 1 letter will be put into the envelope with its correct address?
    c. only 2 letters will be put into the envelope with its correct address?
    d. only 3 letters will be put into the envelope with its correct address?
    e. only 4 letters will be put into the envelope with its correct address?
    f. all 5 letters will be put into the envelope with its correct address?
5. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
6. In a survey on the students' chewing gum preferences, it was found that
    a. 20 like juicy fruit.
    b. 25 like spearmint.
    c. 33 like watermelon.
    d. 12 like spearmint and juicy fruit.
    e. 16 like juicy fruit and watermelon.
    f. 20 like spearmint and watermelon.

g. 5 like all three flavors.
h. 4 like none.

How many students were surveyed?

<u>Bonus: recap problems from previous homeworks – solve the ones you have not yet solved</u>

7. Using the method of mathematical induction, prove the following equality,

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

8. Put the sign $<$, $>$, or $=$, in place of ... below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

9. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2 + \left(4+\frac{1}{4}\right)^2 + \cdots + \left(2^n + \frac{1}{2^n}\right)^2$$

10. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?

11. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

12. Find the following sum,
 a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
 b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n-1) \cdot 3^n$

13. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

14. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

15. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x| \le 1$?

16. Find the coefficient multiplying x^9 after all parentheses are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.

Geometry.

Review the previous classwork notes on the method of coordinates. No new geometry problems: please try solving the unsolved problems from the last homework, which are repeated below.

Problems.

1. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.

2. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).

    a. $|x| = |y|$

    b. $|x|+x = |y|+y$

    c. $|x|/x = |y|/y$

    d. $[y] = [x]$

    e. $\{y\} = \{x\}$

    f. $x^2 - y^2 \ge 0$

g. $x^2 + y^2 \leq 1$

h. $x^2 + 8x = 9 - y^2$

3. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point $P\ (0,2)$ on the Y-axis. Write the formula relating y and x for these points.
4. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
5. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.
6. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. **Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.