

## Geometry.

### Trigonometry homework review.

Solutions to selected homework problems:

- Find the sum of the following series,

$$S_n = \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots + \cos nx$$

(hint: multiply the sum by  $2 \sin x/2$ )

Solution 1: Easy way of summing the trigonometric series is by multiplying and dividing it with  $\sin \frac{x}{2}$ ,

$$\begin{aligned} S_n \frac{\sin \frac{x}{2}}{\sin \frac{x}{2}} &= \frac{\sin \frac{x}{2} (\cos x + \cos 2x + \cdots + \cos nx)}{\sin \frac{x}{2}} = \frac{\sin \frac{x}{2} \cos x + \sin \frac{x}{2} \cos 2x + \cdots + \sin \frac{x}{2} \cos nx}{\sin \frac{x}{2}} = \\ &= \frac{\frac{1}{2} (-\sin \frac{x}{2} + \sin \frac{3x}{2} - \sin \frac{3x}{2} + \sin \frac{5x}{2} - \sin \frac{5x}{2} + \cdots - \sin (n - \frac{1}{2})x + \sin (n + \frac{1}{2})x)}{\sin \frac{x}{2}} = \frac{\frac{1}{2} (-\sin \frac{x}{2} + \sin (n + \frac{1}{2})x)}{\sin \frac{x}{2}} = \\ &= \frac{\cos \frac{(n+1)x}{2} \sin \frac{nx}{2}}{\sin \frac{x}{2}}. \end{aligned}$$

Solution 2. A slightly different and perhaps easier way of summing the above trigonometric series is by adding the expression for  $S_n$ , or  $S_2$ , rearranged from back to front, to itself, as we did when summing the arithmetic series,

$$S_n = \cos x + \cos 2x + \cdots + \cos nx$$

$$S_n = \cos nx + \cos(n-1)x + \cdots + \cos x$$

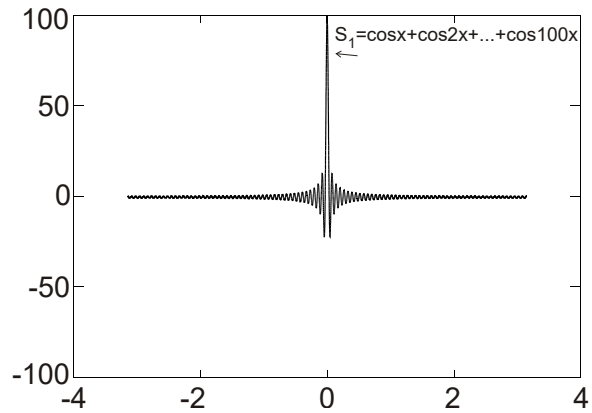
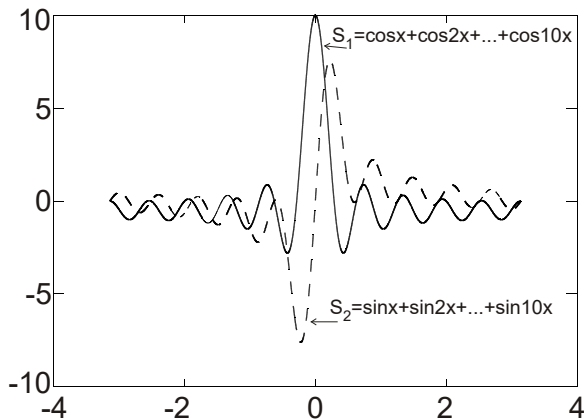
Wherefrom,

$$\begin{aligned} S_n &= \frac{1}{2} ((\cos x + \cos nx) + (\cos 2x + \cos(n-1)x) + \cdots + (\cos nx + \\ &\cos x)) = \cos \frac{(n+1)x}{2} \left( \cos(n-1)\frac{x}{2} + \cos(n-3)\frac{x}{2} + \cdots + \cos(n-1)\frac{x}{2} \right) = \end{aligned}$$

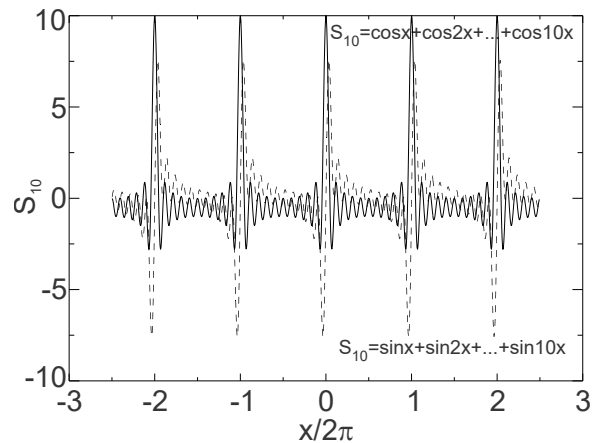
$$\cos \frac{(n+1)x}{2} \frac{(\sin \frac{x}{2} \cos(n-1)\frac{x}{2} + \sin \frac{x}{2} \cos(n-3)\frac{x}{2} + \dots + \sin \frac{x}{2} \cos(n-1)\frac{x}{2})}{\sin \frac{x}{2}} =$$

$$\cos \frac{(n+1)x}{2} \frac{\frac{1}{2}(\sin \frac{nx}{2} - \sin \frac{(n-2)x}{2} + \sin \frac{(n-2)x}{2} - \sin \frac{(n-4)x}{2} + \dots + \sin \frac{nx}{2})}{\sin \frac{x}{2}} = \cos \frac{(n+1)x}{2} \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$

It is interesting to look at a function  $S_n(x)$ .



Behavior of  $S_n(x)$  is intuitively clear. For  $x = 0$ , all terms in the sum are equal to 1, and the sum equals to the number of terms,  $S_n(0) = n$ , while for  $x \neq 0$  it consists of a large number of positive and negative terms, which tend to cancel each other. Note that is periodic with period  $2\pi$ . The above figures only show one period. Here is how 5 periods look.



2. Prove the following equalities:

a.  $\frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} = \cot \frac{\alpha}{2}$

Solution:

$$\frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} = \frac{1 + \cos \alpha}{\sin \alpha} = \frac{2 \cos^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \cot \frac{\alpha}{2}$$

$$\text{b. } \sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$$

Solution:

$$\begin{aligned} \sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) &= \left(\sin\left(\frac{7\pi}{8} - 2\alpha\right) - \sin\left(\frac{9\pi}{8} - 2\alpha\right)\right) \left(\sin\left(\frac{7\pi}{8} - 2\alpha\right) + \sin\left(\frac{9\pi}{8} - 2\alpha\right)\right) \\ &= 2 \cos \frac{2\pi - 4\alpha}{2} \sin\left(-\frac{\pi}{8}\right) 2 \sin \frac{2\pi - 4\alpha}{2} \cos\left(-\frac{\pi}{8}\right) \\ &= -2 \sin(\pi - 2\alpha) \cos(\pi - 2\alpha) 2 \sin\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{8}\right) = \sin 4\alpha \sin \frac{\pi}{4} = \frac{\sin 4\alpha}{\sqrt{2}} \end{aligned}$$

$$\text{c. } (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \sin^2 \frac{\alpha - \beta}{2}$$

Solution:

$$\begin{aligned} (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 &= 4 \sin^2 \frac{\alpha + \beta}{2} \sin^2 \frac{\alpha - \beta}{2} + 4 \cos^2 \frac{\alpha + \beta}{2} \sin^2 \frac{\alpha - \beta}{2} \\ &= 4 \sin^2 \frac{\alpha - \beta}{2} \left( \sin^2 \frac{\alpha + \beta}{2} + \cos^2 \frac{\alpha + \beta}{2} \right) = 4 \sin^2 \frac{\alpha - \beta}{2} \end{aligned}$$

$$\text{d. } \frac{\cot^2 2\alpha - 1}{2 \cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$$

Solution:

$$\begin{aligned} \frac{\cot^2 2\alpha - 1}{2 \cot 2\alpha} - \cos 8\alpha \cot 4\alpha &= \frac{\cos^2 2\alpha - \sin^2 2\alpha}{2 \sin 2\alpha \cos 2\alpha} - \cos 8\alpha \frac{\cos 4\alpha}{\sin 4\alpha} = \cot 4\alpha (1 - \cos 8\alpha) \\ &= \frac{\cos 4\alpha}{\sin 4\alpha} 2 \sin^2 4\alpha = 2 \sin 4\alpha \cos 4\alpha = \sin 8\alpha \end{aligned}$$

$$\text{e. } \sin^6 \alpha + \cos^6 \alpha + 3 \sin^2 \alpha \cos^2 \alpha = 1$$

Solution:

$$\begin{aligned} \sin^6 \alpha + \cos^6 \alpha + 3 \sin^2 \alpha \cos^2 \alpha &= \left(\frac{3 \sin \alpha - \sin 3\alpha}{4}\right)^2 + \left(\frac{\cos 3\alpha + 3 \cos \alpha}{4}\right)^2 + \frac{3}{4} \sin^2 2\alpha \\ &= \frac{1}{16} (9 \sin^2 \alpha + \sin^2 3\alpha - 6 \sin \alpha \sin 3\alpha + \cos^2 3\alpha + 9 \cos^2 \alpha + 6 \cos 3\alpha \cos \alpha + 12 \sin^2 2\alpha) \\ &= \frac{1}{16} (10 + 6(\cos 3\alpha \cos \alpha - \sin 3\alpha \sin \alpha) + 6(1 - \cos 4\alpha)) \\ &= \frac{1}{16} (10 + 6 \cos 4\alpha + 6(1 - \cos 4\alpha)) = 1 \end{aligned}$$

$$f. \frac{\sin 6\alpha + \sin 7\alpha + \sin 8\alpha + \sin 9\alpha}{\cos 6\alpha + \cos 7\alpha + \cos 8\alpha + \cos 9\alpha} = \tan \frac{15\alpha}{2}$$

Solution:

$$\begin{aligned} \frac{\sin 6\alpha + \sin 7\alpha + \sin 8\alpha + \sin 9\alpha}{\cos 6\alpha + \cos 7\alpha + \cos 8\alpha + \cos 9\alpha} &= \frac{(\sin 6\alpha + \sin 9\alpha) + (\sin 8\alpha + \sin 7\alpha)}{(\cos 6\alpha + \cos 9\alpha) + (\cos 8\alpha + \cos 7\alpha)} = \\ \frac{2 \sin \frac{15}{2}\alpha \cos \frac{3}{2}\alpha + 2 \sin \frac{15}{2}\alpha \cos \frac{1}{2}\alpha}{2 \cos \frac{15}{2}\alpha \cos \frac{3}{2}\alpha + 2 \cos \frac{15}{2}\alpha \cos \frac{1}{2}\alpha} &= \frac{\sin \frac{15}{2}\alpha \cos \frac{3}{2}\alpha + \cos \frac{1}{2}\alpha}{\cos \frac{15}{2}\alpha \cos \frac{3}{2}\alpha + \cos \frac{1}{2}\alpha} = \tan \frac{15}{2}\alpha \end{aligned}$$

$$g. \sin^6 \alpha + \cos^6 \alpha = \frac{5 + 3 \cos 4\alpha}{8}$$

Solution:

$$\begin{aligned} \sin^6 \alpha + \cos^6 \alpha &= \left( \frac{3 \sin \alpha - \sin 3\alpha}{4} \right)^2 + \left( \frac{\cos 3\alpha + 3 \cos \alpha}{4} \right)^2 = \frac{1}{16} (9 \sin^2 \alpha + \sin^2 3\alpha - \\ 6 \sin \alpha \sin 3\alpha + \cos^2 3\alpha + 9 \cos^2 \alpha + 6 \cos 3\alpha \cos \alpha) &= \frac{1}{16} (10 + \\ 6(\cos 3\alpha \cos \alpha - \sin 3\alpha \sin \alpha)) &= \frac{1}{16} (10 + 6 \cos 4\alpha) = \frac{5 + 3 \cos 4\alpha}{8} \end{aligned}$$

$$h. 16 \sin^5 \alpha - 20 \sin^3 \alpha + 5 \sin \alpha = \sin 5\alpha$$

Solution:

$$\begin{aligned} \sin 5\alpha &= \sin \alpha \cos 4\alpha + \cos \alpha \sin 4\alpha = \sin \alpha (2 \cos^2 2\alpha - 1) + \\ \cos \alpha 2 \sin 2\alpha \cos 2\alpha &= \sin \alpha (2(1 - 2 \sin^2 \alpha)^2 - 1) + 4 \sin \alpha \cos^2 \alpha (1 - \\ 2 \sin^2 \alpha) &= \sin \alpha (1 - 8 \sin^2 \alpha + 8 \sin^4 \alpha + 4(1 - \sin^2 \alpha)(1 - \\ 2 \sin^2 \alpha)) &= \sin \alpha (5 - 20 \sin^2 \alpha + 16 \sin^4 \alpha) \end{aligned}$$

$$i. \frac{\cos 64^\circ \cos 4^\circ - \cos 86^\circ \cos 26^\circ}{\cos 71^\circ \cos 41^\circ - \cos 49^\circ \cos 19^\circ}$$

Solution:

$$\frac{\cos 64^\circ \cos 4^\circ - \cos 86^\circ \cos 26^\circ}{\cos 71^\circ \cos 41^\circ - \cos 49^\circ \cos 19^\circ} = \frac{\cos 60^\circ + \cos 68^\circ - \cos 60^\circ - \cos 112^\circ}{\cos 30^\circ + \cos 112^\circ - \cos 30^\circ - \cos 68^\circ} = \frac{\cos 68^\circ - \cos 112^\circ}{\cos 112^\circ - \cos 68^\circ} = -1$$

$$j. \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$$

Solution: denote  $x = 20^\circ$ ,  $\cos 3x = \cos 60^\circ = \frac{1}{2}$ ,  $\cos 6x = \cos 120^\circ = -\frac{1}{2}$ ,

$$\begin{aligned} \sin x \sin 2x \sin 3x \sin 4x &= \sin x \sin 3x \sin 2x \sin 4x = \frac{1}{2}(\cos 2x - \cos 4x) \frac{1}{2}(\cos 2x - \cos 6x) \\ &= \frac{1}{2}(\cos 2x - (2\cos^2 2x - 1)) \left(\cos 2x + \frac{1}{2}\right) \\ &= \frac{1}{2}(\cos^2 2x - 2\cos^3 2x + \cos 2x + \frac{1}{2}\cos 2x - \cos^2 2x + \frac{1}{2}) = \\ &= \frac{1}{4}(1 - 4\cos^3 2x + 3\cos 2x) = \frac{1}{4}(1 - \cos 6x) = \frac{3}{16} \end{aligned}$$

k.  $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$

Solution:

$$\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4 \frac{\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ}{2\sin 10^\circ \cos 10^\circ} = 4 \frac{\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ}{2\sin 10^\circ \cos 10^\circ} = 4 \frac{\sin(30^\circ - 10^\circ)}{\sin 20^\circ} = 4$$

## Trigonometry homework review. Part 2.

3. Simplify the following expressions:

l.  $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

Solution:

$$\begin{aligned} \sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right) &= \left(\sin\left(\frac{\alpha}{2} + 2\beta\right) - \sin\left(\frac{\alpha}{2} - 2\beta\right)\right) \left(\sin\left(\frac{\alpha}{2} + 2\beta\right) + \sin\left(\frac{\alpha}{2} - 2\beta\right)\right) \\ &= 2\cos\frac{\alpha}{2}\sin 2\beta \cdot 2\sin\frac{\alpha}{2}\cos 2\beta = \sin \alpha \sin 4\beta. \end{aligned}$$

m.  $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

Solution:

$$\begin{aligned} 2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1 &= \cos 6\alpha + 1 + \sqrt{3}\sin 6\alpha - 1 \\ &= 2\left(\frac{1}{2}\cos 6\alpha + \frac{\sqrt{3}}{2}\sin 6\alpha\right) = 2\left(\sin\frac{\pi}{6}\cos 6\alpha + \cos\frac{\pi}{6}\sin 6\alpha\right) \\ &= 2\sin\left(\frac{\pi}{6} + 6\alpha\right) \end{aligned}$$

n.  $\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha$

Solution:

$$\begin{aligned} \cos^4 2\alpha - 6 \cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha &= (\cos^2 2\alpha - \sin^2 2\alpha)^2 - \\ 4 \cos^2 2\alpha \sin^2 2\alpha &= \cos^2 4\alpha - \sin^2 4\alpha = \cos 8\alpha \end{aligned}$$

o.  $\sin^2(135^\circ - 2\alpha) - \sin^2(210^\circ - 2\alpha) - \sin 195^\circ \cos(165^\circ - 4\alpha)$ .

Solution:

$$\begin{aligned} \sin^2(135^\circ - 2\alpha) - \sin^2(210^\circ - 2\alpha) - \sin 15^\circ \cos(165^\circ - 4\alpha) &= \\ \sin^2(45^\circ + 2\alpha) - \sin^2(2\alpha - 30^\circ) - \sin 15^\circ \cos(15^\circ + 4\alpha) &= (\sin(45^\circ + \\ 2\alpha) - \sin(2\alpha - 30^\circ))(\sin(45^\circ + 2\alpha) + \sin(2\alpha - 30^\circ)) - \sin 15^\circ \cos(15^\circ + \\ 4\alpha) &= 2 \cos \frac{15^\circ + 4\alpha}{2} \sin \frac{75^\circ}{2} 2 \sin \frac{15^\circ + 4\alpha}{2} \cos \frac{75^\circ}{2} - \sin 15^\circ \cos(15^\circ + 4\alpha) = \\ \sin 75^\circ \sin(15^\circ + 4\alpha) - \sin 15^\circ \cos(15^\circ + 4\alpha) &= \sin(15^\circ + 4\alpha) \cos 15^\circ - \\ \cos(15^\circ + 4\alpha) \sin 15^\circ &= \sin(4\alpha) \end{aligned}$$

p.  $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

Solution:

$$\begin{aligned} \frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha} &= \frac{2 \sin 8\alpha \sin 6\alpha - 2 \sin 8\alpha \sin 2\alpha}{2 \sin 8\alpha \cos 2\alpha + 2 \sin 8\alpha \cos 6\alpha} \\ &= \frac{\sin 6\alpha - \sin 2\alpha}{\cos 2\alpha + \cos 6\alpha} = \frac{2 \cos 4\alpha \sin 2\alpha}{2 \cos 4\alpha \cos 2\alpha} = \tan 2\alpha \end{aligned}$$

4. Let  $A, B$  and  $C$  be angles of a triangle. Prove that

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

Solution:

$$\begin{aligned} \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} &= \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \left( \frac{\pi}{2} - \frac{A+B}{2} \right) + \\ \tan \left( \frac{\pi}{2} - \frac{A+B}{2} \right) \tan \frac{A}{2} &= \tan \frac{A}{2} \tan \frac{B}{2} + \cot \frac{A+B}{2} \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) = \tan \frac{A}{2} \tan \frac{B}{2} + \\ \cot \frac{A+B}{2} \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \sin \frac{B}{2} \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} &= \frac{\sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} + \cot \frac{A+B}{2} \frac{\sin \frac{A+B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = \frac{\sin \frac{A}{2} \sin \frac{B}{2} + \cos \frac{A+B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = 1 \end{aligned}$$