

## Algebra.

Solutions to some homework problems.

1. **Problem.** Write the first few terms in the following sequence ( $n \geq 1$ ),

$$n \text{ fractions} \left\{ \begin{array}{l} \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \\ \dots + \frac{1}{1+x} \end{array} \right. = f_n$$

- Try guessing the general formula of this fraction for any  $n$ .
- Using mathematical induction, try proving the formula you guessed.

**Solution.**  $n = 1: f_1 = \frac{1}{1+x}; n = 2: f_2 = \frac{1}{1 + \frac{1}{1+x}} = \frac{1+x}{2+x}; n = 3, f_3 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1+x}}} = \frac{2+x}{3+2x}; n = 4, f_4 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+x}}}} = \frac{3+2x}{5+3x}; f_5 = \frac{5+3x}{8+5x}; \dots$

From the definition, we can write the recurrence,  $f_{n+1} = \frac{1}{1+f_n}$ . We note, that if  $f_n = \frac{a_n+b_nx}{c_n+d_nx}$ , then  $f_{n+1} = \frac{c_n+d_nx}{(a_n+c_n)+(b_n+d_n)x}$ . Hence, in each next term,  $f_{n+1}$ , in the sequence, the numerator is equal to the denominator of the previous term,  $f_n$ , while the numbers in the denominator are the sums of the corresponding numbers in the numerator and the denominator of the previous term,  $f_n$ , thus forming the Fibonacci sequence,  $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$ . We can thus guess,

- $n$  fractions:  $f_1 = \frac{1}{1+x}, f_n = \frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}, n > 1$
- Base:  $f_2 = \frac{1+x}{1+2x}$

Induction: Using the recurrence implied in the definition,

$$f_{n+1} = \frac{1}{1+f_n} = \frac{1}{1 + \frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}} = \frac{F_{n+1}+F_nx}{F_{n+1}+F_n+F_nx+F_{n-1}x} = \frac{F_{n+1}+F_nx}{F_{n+2}+F_{n+1}x}.$$

2. **Problem.** Can you prove that,

a.

$$\frac{3+\sqrt{17}}{2} = 3 + \frac{2}{3+\frac{2}{3+\frac{2}{3+\dots}}}$$

b.  $1 = 3 - \frac{2}{3-\frac{2}{3-\frac{2}{3-\dots}}}$  ?

c.

$$\frac{4}{2+\frac{4}{2+\frac{4}{2+\dots}}} = 1 + \frac{1}{4+\frac{1}{4+\frac{1}{4+\dots}}}$$

Find these numbers?

**Solution.** Consider a general continued fraction,

$$x = a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \dots}}}$$

If a number exists, which is equal to the above infinite continued fraction, then it must satisfy the equation,  $x = a + \frac{b}{x} \Leftrightarrow x^2 - ax - b = 0$

$\Leftrightarrow x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + b}$ . If  $a$  and  $b$  are positive, then  $x$  must also be

positive, so  $x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + b}$ .

a. Following the above argument with  $a = 3, b = 2$ , we obtain,  $x = \frac{3}{2} +$

$$\sqrt{\left(\frac{3}{2}\right)^2 + 2} = \frac{3+\sqrt{17}}{2}$$

b. In this case,  $a = 3$ , but  $b = -2$  is negative. Applying the above considerations naively, we obtain,  $x = 3 - \frac{2}{x} \Leftrightarrow x^2 - 3x + 2 = 0$   
 $\Leftrightarrow (x - 1)(x - 2) = 0$ , i.e. there are two equally “legitimate” answers,  $x = 1$ , or  $x = 2$ . What this means, is that assumption that there exist unique number encoded by the given infinite continued fraction is wrong: there exists no such number! In fact, this can also be understood by looking at finite truncations approximating this

continued fraction. If the continued fraction is truncated after subtracting 2 and before division by 3, then it is equal to 1,

$$3 - \frac{2}{3-2} = 1, 3 - \frac{2}{3 - \frac{2}{3-2}} = 1, \dots$$

If, on the other hand, the truncation is after division by 3 and before subtracting 2, then we obtain a sequence of numbers approaching 2,

$$3 - \frac{2}{3} = 2\frac{1}{3}, 3 - \frac{2}{3 - \frac{2}{3}} = 2\frac{1}{7}, 3 - \frac{2}{3 - \frac{2}{3 - \frac{2}{3}}} = 2\frac{1}{15}, \dots$$

c. Denote

$$x = \frac{4}{2 + \frac{4}{2 + \frac{4}{2 + \dots}}} = \frac{4}{2 + x}$$

Then,  $x^2 + 2x - 4 = 0 \Leftrightarrow x = -1 \pm \frac{\sqrt{5}}{2}$ , and  $x > 0$ . Hence,  $x = -1 + \frac{\sqrt{5}}{2}$ .

Similarly, denote

$$y = \frac{1}{4 + \frac{1}{4 + \frac{1}{4 + \dots}}} = \frac{1}{4 + y}$$

Then,  $y^2 + 4y - 1 = 0 \Leftrightarrow y = -2 \pm \frac{\sqrt{5}}{2}$ , and  $y > 0$ . Hence,  $y = -2 + \frac{\sqrt{5}}{2}$ , and  $1 + y = -1 + \frac{\sqrt{5}}{2} = x$ .