

October 19, 2025

## Algebra.

### Principle of Mathematical Induction (continued).

#### Newton's binomial.

The **Newton's binomial** is an expression representing the simplest  $n$ -th degree factorized polynomial of two variables,  $P_n(x, y) = (x + y)^n$  in the form of the polynomial summation (i.e. expanding the brackets),

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n, \quad (1a)$$

$$(x + y)^n = C_n^0 x^n + C_n^1 x^{n-1}y + C_n^2 x^{n-2}y^2 + \cdots + C_n^k x^{n-k}y^k + \cdots + C_n^{n-1} x y^{n-1} + C_n^n y^n. \quad (1b)$$

For  $n = 1, 2, 3, \dots$ , these are familiar expressions,

$$(x + y) = x + y,$$

$$(x + y)^2 = x^2 + 2xy + y^2,$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

etc.

The Newton's binomial formula could be established either by directly expanding the brackets or proven using the mathematical induction.

**Exercise.** Prove the Newton's binomial using the mathematical induction.

**Induction basis.** For  $n = 1$  the statement is a true equality,  $(x + y)^1 = C_1^0 x + C_1^1 y$ . We can also easily prove that it holds for  $n = 2$ . Indeed,  $(x + y)^2 = C_2^0 x^2 + C_2^1 xy + C_2^2 y^2$ .

**Induction hypothesis.** Suppose the equality holds for some  $n \in N$ , that is,

$$(x + y)^n = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \dots + C_n^k x^{n-k} y^k + \dots + C_n^{n-1} x y^{n-1} + C_n^n y^n$$

**Induction step.** We have to prove that it then also holds for the next integer,  $n + 1$ ,

$$(x + y)^{n+1} = C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \dots + C_{n+1}^k x^{n+1-k} y^k + \dots + C_{n+1}^n x y^n + C_{n+1}^{n+1} y^{n+1}$$

**Proof.**  $(x + y)^{n+1} = (x + y)^n (x + y) =$

$$(C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \dots + C_n^k x^{n-k} y^k + \dots + C_n^{n-1} x y^{n-1} + C_n^n y^n)(x + y) =$$

$$C_n^0 x^{n+1} + C_n^1 x^n y + C_n^2 x^{n-1} y^2 + \dots + C_n^k x^{n-k+1} y^k + \dots + C_n^{n-1} x^2 y^{n-1} + C_n^n x y^n + C_n^0 x^n y + C_n^1 x^{n-1} y^2 + C_n^2 x^{n-2} y^3 + \dots + C_n^k x^{n-k} y^{k+1} + \dots + C_n^{n-1} x y^n + C_n^n y^{n+1} =$$

$$C_n^0 x^{n+1} + (C_n^1 + C_n^0) x^n y + (C_n^2 + C_n^1) x^{n-1} y^2 + \dots + (C_n^k + C_n^{k-1}) x^{n-k+1} y^k + \dots + (C_n^n + C_n^{n-1}) x y^n + C_n^n y^{n+1} =$$

$$C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \dots + C_{n+1}^k x^{n+1-k} y^k + \dots + C_{n+1}^n x y^n + C_{n+1}^{n+1} y^{n+1},$$

Where we have used the property of binomial coefficients,  $C_n^k + C_n^{k-1} = C_{n+1}^k$ .  
□

### Recap: Properties of binomial coefficients

Binomial coefficients are defined by

$$C_n^k = {}_n C_k = \binom{n}{k} = \frac{n!}{k! (n - k)!}$$

Binomial coefficients have clear and important combinatorial meaning.

- There are  $\binom{n}{k}$  ways to choose  $k$  elements from a set of  $n$  elements.
- There are  $\binom{n + k - 1}{k}$  ways to choose  $k$  elements from a set of  $n$  if repetitions are allowed.

- There are  $\binom{n+k}{k}$  strings containing  $k$  ones and  $n$  zeros.
- There are  $\binom{n+1}{k}$  strings consisting of  $k$  ones and  $n$  zeros such that no two ones are adjacent.

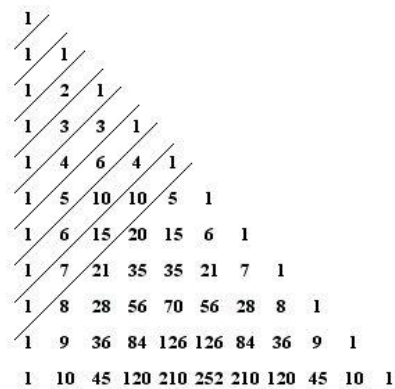
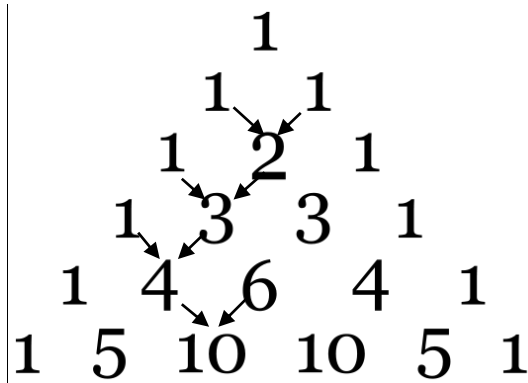
They satisfy the following identities,

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_{n+1}^k = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\sum_{k=0}^n C_n^k = \sum_{k=0}^n \binom{n}{k} = 2^n$$

### Patterns in the Pascal triangle



|                                     |                                                    |
|-------------------------------------|----------------------------------------------------|
| $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ | Fibonacci numbers (sum of the "shallow" diagonals: |
|-------------------------------------|----------------------------------------------------|

**Exercise.** Find the sum of the top  $n$  rows in the Pascal triangle,

$$\sum_{m=0}^n (\sum_{k=0}^m C_m^k) = 2^{n+1} - 1.$$

### Review of selected homework problems.

**Problem 4.** Using mathematical induction, prove that

$$\text{a. } P_n: \sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Solution.**

Basis:  $P_1: \sum_{k=1}^1 k^2 = 1 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6}$

Induction:  $P_n \Rightarrow P_{n+1}$ , where  $P_{n+1}: \sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

Proof:  $\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^n k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)}{6} (n(2n+1) + 6n+6) = \frac{(2n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$ ,

where we used the induction hypothesis,  $P_n$ , to replace the sum of the first  $n$  terms with a formula given by  $P_n$ .  $\square$

$$\text{b. } P_n: \sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

**Solution.**

Basis:  $P_1: \sum_{k=1}^1 k^3 = 1 = \left[ \frac{1(1+1)}{2} \right]^2$

Induction:  $P_n \Rightarrow P_{n+1}$ , where  $P_{n+1}: \sum_{k=1}^{n+1} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n+1)^3 = \left[ \frac{(n+1)(n+2)}{2} \right]^2$

Proof:  $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3 = \left[ \frac{n(n+1)}{2} \right]^2 + (n+1)^3 = \left[ \frac{(n+1)}{2} \right]^2 (n^2 + 4n + 4) = \left[ \frac{(n+1)(n+2)}{2} \right]^2$ , where we used the induction hypothesis,  $P_n$ , to replace the sum of the first  $n$  terms with a formula given by  $P_n$ .  $\square$

$$\text{c. } P_n: \sum_{k=1}^n \frac{1}{k^2+k} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

**Solution.**

Basis:  $P_1: \sum_{k=1}^1 \frac{1}{k^2+k} = \frac{1}{2} = \frac{1}{1+1}$

Induction:  $P_n \Rightarrow P_{n+1}$ , where  $P_{n+1}: \sum_{k=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+2}$

Proof:  $\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=0}^n \frac{1}{k^2+k} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ .  $\square$

e.  $P_n: \forall n, \exists k, 5^n + 3 = 4k$

**Solution.**

Basis:  $P_1: n = 1, \exists k, 5^1 + 3 = 8 = 4k \Leftrightarrow k = 2$

Induction:  $P_n \Rightarrow P_{n+1}$ , where  $P_{n+1}: \forall n, \exists q, 5^{n+1} + 3 = 4q$

Proof:  $5^{n+1} + 3 = 5 \cdot 5^n + 3 = 5 \cdot (4k - 3) + 3 = 5 \cdot 4k - 12 = 4 \cdot (5k - 3)$ .

Where we used the induction hypothesis,  $P_n$ , to replace  $5^n$  with a formula,  $5^n = 4k - 3$ , given by  $P_n$ .  $\square$

e.  $P_n: \forall n \geq 2, \forall x > -1, (1+x)^n \geq 1+nx$

**Solution.**

Basis:  $P_2: \forall x > -1, n = 2, (1+x)^2 = 1 + 2x + x^2 \geq 1 + 2x$

Induction:  $P_n \Rightarrow P_{n+1}$ , where  $P_{n+1}: \forall n \geq 2, \forall x > -1, (1+x)^{n+1} \geq 1 + (n+1)x$

Proof:  $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1 + (n+1)x + x^2 \geq 1 + (n+1)x$ .  $\square$