

Algebra.

Trigonometric form of complex numbers. Geometric interpretation.

Let us consider complex numbers with the absolute value of 1,

$$z_1 = x_1 + iy_1, |z_1|^2 = z_1 \bar{z}_1 = x_1^2 + y_1^2 = 1.$$

There is an obvious one-to-one correspondence between such numbers and points $Z_1(x_1, y_1)$ on a circle of unit radius. Hence, we can express such numbers in terms of an angle, φ , parameterizing points on the unit circle,

$$z_1 = x_1 + iy_1 = \cos \varphi + i \sin \varphi.$$

More generally, any complex number, $z = x + iy$, whose absolute value is $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = r$, can be written in the trigonometric form as,

$$z = x + iy = r(\cos \varphi + i \sin \varphi).$$

Geometrically, it is represented by a point $Z(x, y)$ on a circle of radius $r = |z|$. Position of this point is specified by an angle, φ , which is conventionally measured counterclockwise from the positive direction of the X -axis. Angle φ is called the argument of the complex number z and is denoted $\varphi = \text{Arg}(z)$. Thus, instead of describing a complex number by its real and imaginary part, i.e. its coordinates, (x, y) , we can describe it by its magnitude and argument (polar coordinates), (r, φ) , where $r \geq 0$ and $0 \leq \varphi = \text{Arg}(z) < 360^\circ$.

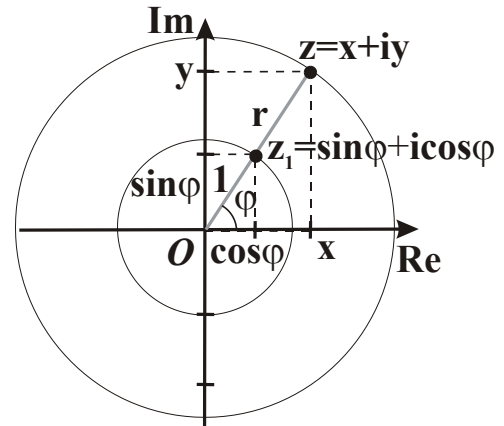
It is now easy to prove the following important property of the multiplication of complex numbers.

Theorem. When we multiply two complex numbers, magnitudes multiply and arguments add,

$$|z_1 z_2| = |z_1| |z_2|, \text{Arg}(z_1 z_2) = (\text{Arg}(z_1) + \text{Arg}(z_2)) \bmod 360^\circ.$$

Proof. Let $\text{Arg}(z_1) = \varphi_1$ and $\text{Arg}(z_2) = \varphi_2$, so $z_1 = |z_1|(\cos \varphi_1 + i \sin \varphi_1)$ and $z_2 = |z_2|(\cos \varphi_2 + i \sin \varphi_2)$. Perform the multiplication directly,

$$z_1 z_2 = |z_1|(\cos \varphi_1 + i \sin \varphi_1) |z_2|(\cos \varphi_2 + i \sin \varphi_2) =$$



$$|z_1||z_2|(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 + i(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2)) \\ = |z_1||z_2|(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$$

Complex numbers whose arguments would differ by multiples of 360° are identical and correspond to the same point on the complex plane. Hence, the argument is computed *mod* 360° , ensuring that $0 \leq \varphi_1 + \varphi_2 < 360^\circ$.

Theorem. Multiplication of a complex number, $z = x + iy = r(\cos \varphi + i \sin \varphi)$, by a complex number of unit magnitude and argument ψ ,

$$z_\psi = x_\psi + iy_\psi = \cos \psi + i \sin \psi,$$

corresponds to a counterclockwise rotation of the point, $Z(x, y)$, on the complex plane, by an angle ψ ,

$$|zz_\psi| = r, \text{ Arg}(zz_\psi) = \varphi + \psi.$$

Proof. Indeed, perform the multiplication directly,

$$zz_\psi = (x + iy)(x_\psi + iy_\psi) = r(\cos \varphi + i \sin \varphi)(\cos \psi + i \sin \psi) \\ = r(\cos \varphi \cos \psi - \sin \varphi \sin \psi + i(\sin \varphi \cos \psi + \cos \varphi \sin \psi)) \\ = r(\cos(\varphi + \psi) + i \sin(\varphi + \psi))$$

It is clear that multiplication by a complex number with magnitude r' and argument ψ is equivalent to the combination of multiplication by a number of unit magnitude and argument ψ , and by the real number r' .

Theorem. Multiplication of a complex number, $z = x + iy$, by a complex number with magnitude r' and argument ψ ,

$$w = r'(\cos \psi + i \sin \psi),$$

results in a point on the complex plane, which is obtained from the point $Z(x, y)$ by the combination of a rotation by angle ψ and a homothety (rescaling) with scale factor r' .

Multiplication of all complex numbers by a complex number $w = r'(\cos \psi + i \sin \psi)$ is a transformation of the complex plane, which maps complex plane on itself. Identifying multiplication by a complex number with such transformation, we can state the following.

Theorem. Multiplication by a complex number with magnitude r' and argument ψ , $w = r'(\cos \psi + i \sin \psi)$, is a combination of rotation by angle ψ and homothety (rescaling) with scale factor r' .

De Moivre's formula.

Theorem. The formula named after Abraham de Moivre states that for any complex number, $z = x + iy = r(\cos \varphi + i \sin \varphi)$, and for any integer $n \in \mathbb{N}$,

$$z^n = (r(\cos \varphi + i \sin \varphi))^n = r^n(\cos n\varphi + i \sin n\varphi)$$

Proof 1 (Mathematical induction).

1. Base case, $n = 1$: $z^1 = r(\cos \varphi + i \sin \varphi)$ is true.
2. $I(n) \Rightarrow I(n + 1)$. Assume $z^n = r^n(\cos n\varphi + i \sin n\varphi)$ is true. Then,

$$\begin{aligned} z^{n+1} &= z \cdot z^n = r(\cos \varphi + i \sin \varphi) \cdot (r(\cos \varphi + i \sin \varphi))^n \\ &= r(\cos \varphi + i \sin \varphi)r^n(\cos n\varphi + i \sin n\varphi) \\ &= r^{n+1}(\cos \varphi \cos n\varphi - \sin \varphi \sin n\varphi \\ &\quad + i(\sin \varphi \cos n\varphi + \cos \varphi \sin n\varphi)) \\ &= r^{n+1}(\cos(n + 1)\varphi + i \sin(n + 1)\varphi) \end{aligned}$$

Proof 2 (Geometrical).

$$z^n = (r(\cos \varphi + i \sin \varphi))^n = r(\cos \varphi + i \sin \varphi)r(\cos \varphi + i \sin \varphi) \dots r(\cos \varphi + i \sin \varphi)$$

By property of the multiplication of complex numbers, absolute values multiply, while arguments add. Therefore,

$$|z^n| = |z|^n = r^n, \text{ and } Arg(z^n) = nArg(z) = n\varphi, \text{ wherefrom it follows that } z^n = r^n(\cos n\varphi + i \sin n\varphi).$$

n-th root.

The formula of de Moivre allows us to compute n -th root of a complex number. Suppose we want to solve the equation,

$$w^n = z$$

where $w, z \in \mathbb{C}$, so w is the n -th root of z . According to de Moivre's formula, if $w = |w|(\cos \psi + i \sin \psi)$, then $w^n = |w|^n(\cos n\psi + i \sin n\psi)$. Denoting $z = r(\cos \varphi + i \sin \varphi)$, we can rewrite the equation as,

$$w^n = |w|^n(\cos n\psi + i \sin n\psi) = r(\cos \varphi + i \sin \varphi)$$

One obvious solution is $r = |w|^n$ and $\varphi = n\psi$, $w = \sqrt[n]{r} \left(\cos \frac{\varphi}{n} + i \sin \frac{\varphi}{n} \right)$.

However, because $\varphi = \text{Arg}(z) = \text{Arg}(w^n)$ and $\psi = \text{Arg}(w)$ are determined modulo 360° (2π radians), there are other solutions, too, satisfying the above equation, such as $w = \sqrt[n]{r} \left(\cos \frac{\varphi + 2\pi}{n} + i \sin \frac{\varphi + 2\pi}{n} \right)$. Generally, we must have $r = |w|^n$, and $\varphi = \text{Arg}(z) = \text{Arg}(w^n) \bmod 360^\circ = n\text{Arg}(w) \bmod 360^\circ = n\psi \bmod 360^\circ$. Altogether, there are n solutions,

$$w = \sqrt[n]{r} \left(\cos \frac{\varphi + 2\pi k}{n} + i \sin \frac{\varphi + 2\pi k}{n} \right), 0 \leq k < n$$

This is a special case of the following extremely important result, called the fundamental theorem of algebra.

Theorem. Any polynomial with complex coefficients of degree n has exactly n roots (counting with multiplicities).

There is no simple proof of this theorem (and, in fact, no purely algebraic proof: all the known proofs use some geometric arguments).

In particular, since any polynomial with real coefficients can be considered as a special case of a polynomial with complex coefficients, this shows that any real polynomial of degree n has exactly n complex roots.