

## Algebra.

### Polynomials and factorization.

**Polynomial** is an expression containing variables denoted by some letters, and combined using addition, multiplication and numbers. General form of the  $n$ -th degree polynomial of one variable  $x$  is,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x^1 + a_0. \quad (1)$$

This includes quadratic polynomial for  $n = 2$ , cubic for  $n = 3$ , etc. The general form for the case of more than one variable is quite complex. For example,

$$P_n(x, y) = a_{n,0} x^n + a_{n-1,0} x^{n-1} + \cdots + a_{1,0} x^1 + a_{0,0} + a_{n-1,1} x^{n-1} y + \cdots + a_{1,1} x y + a_{0,1} y + \cdots$$

One should distinguish variables, such as  $x$  and  $y$ , which can take any real values, and the coefficients denoted here by  $a_n$ , etc, which are just fixed numbers defining a particular polynomial.

We consider only polynomials with one variable. The number,  $n$ , which is the highest power of  $x$  appearing (with non-zero coefficient) in the expression of a polynomial  $P$ , is called degree of  $P$  and often denoted  $\deg(P)$ .

One can add, subtract, and multiply polynomials in the obvious way. It is easy to see that for a product of two polynomials,  $P$  and  $Q$ ,

$$\deg(PQ) = \deg(P) + \deg(Q)$$

However, in general one cannot divide polynomials: expression  $\frac{x^3+3}{x^2+x-1}$  is not a polynomial. However, much like with integers, there is "division with remainder" for polynomials, also known as "long division".

### Polynomial division transformation

**Theorem.** Let  $D(x)$  be a polynomial with  $\deg(D) > 0$  (i.e.,  $D$  is not a constant). Then any polynomial  $P(x)$  can be uniquely written in the form

$$P(x) = D(x)Q(x) + R(x)$$

where  $Q(x)$ ,  $R(x)$  are polynomials, and  $\deg(R) < \deg(D)$ . The polynomial  $R(x)$  is called the remainder upon division of  $P(x)$  by  $D(x)$ .

Polynomial division allows for a polynomial to be written in a divisor-quotient form, which is often advantageous. Consider polynomials  $P(x)$ ,  $D(x)$  where  $\deg(D) < \deg(P)$ . Then, for some quotient polynomial  $Q(x)$  and remainder polynomial  $R(x)$  with  $\deg(R) < \deg(D)$ ,

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \Leftrightarrow P(x) = D(x)Q(x) + R(x)$$

This rearrangement is known as the **division transformation** and derives from the corresponding arithmetical identity.

**Polynomial long division algorithm** for dividing a polynomial by another polynomial of the same or lower degree, is a generalized version of the familiar arithmetic technique called long division. It can be done easily by hand, because it separates an otherwise complex division problem into smaller ones.

### **Example**

Find  $\frac{x^3 - 12x^2 - 42}{x - 3}$ .

The problem is written like this:

$$\frac{x^3 - 12x^2 + 0x - 42}{x - 3}$$

The quotient and remainder can then be determined as follows:

1. Divide the first term of the numerator by the highest term of the denominator (meaning the one with the highest power of  $x$ , which in this case is  $x$ ). Place the result above the bar ( $x^3 \div x = x^2$ ).

$$x - 3 \overline{)x^3 - 12x^2 + 0x - 42}$$

2. Multiply the denominator by the result just obtained (the first term of the eventual quotient). Write the result under the first two terms of the numerator ( $x^2 \cdot (x - 3) = x^3 - 3x^2$ ).

$$\begin{array}{r} x^2 \\ x - 3 \overline{)x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \end{array}$$

3. Subtract the product just obtained from the appropriate terms of the original numerator (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ( $(x^3 - 12x^2) - (x^3 - 3x^2) = -12x^2 + 3x^2 = -9x^2$ ) Then, "bring down" the next term from the numerator.

$$\begin{array}{r} x^2 \\ x - 3 \overline{)x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \\ -9x^2 + 0x \end{array}$$

4. Repeat the previous three steps, except this time use the two terms that have just been written as the numerator.

$$\begin{array}{r} x^2 - 9x \\ x - 3 \overline{)x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \\ -9x^2 + 0x \\ \underline{-9x^2 + 27x} \\ -27x - 42 \end{array}$$

5. Repeat step 4. This time, there is nothing to "pull down".

$$\begin{array}{r}
 x^2 - 9x - 27 \\
 x - 3 \overline{) x^3 - 12x^2 + 0x - 42} \\
 \underline{x^3 - 3x^2} \phantom{+ 0x - 42} \\
 -9x^2 + 0x \phantom{- 42} \\
 \underline{-9x^2 + 27x} \phantom{- 42} \\
 -27x - 42 \\
 \underline{-27x + 81} \\
 -123
 \end{array}$$

6. The polynomial above the bar is the quotient, and the number left over ( $-123$ ) is the remainder.

$$\frac{x^3 - 12x^2 - 42}{x - 3} = \underbrace{x^2 - 9x - 27}_{q(x)} - \underbrace{\frac{123}{x - 3}}_{r(x)/g(x)}$$

The long division algorithm for arithmetic can be viewed as a special case of the above algorithm, in which the variable  $x$  is replaced by the specific number 10.

**Little Bézout's (polynomial remainder) theorem. Factoring polynomials.**

**Theorem.** The remainder of a polynomial  $P(x)$  divided by a linear divisor  $(x - a)$  is equal to  $P(a)$ .

The polynomial remainder theorem follows from the definition of polynomial long division; denoting the divisor, quotient and remainder by, respectively,  $G(x)$ ,  $Q(x)$ , and  $R(x)$ , polynomial long division gives a solution of the equation

$$P(x) = Q(x)G(x) + R(x)$$

where the degree of  $R(x)$  is less than that of  $G(x)$ . If we take  $G(x) = x - a$  as the divisor, giving the degree of  $R(x)$  as 0, i.e.  $R(x) = r$ ,

$$P(x) = Q(x)(x - a) + r. \tag{2}$$

Here  $r$  is a number. Setting  $x = a$ , we obtain  $P(a) = r$ .

## Roots of polynomials.

**Definition 1.** A number  $a \in \mathbb{R}$  is called a **root** of polynomial  $P(x)$  if  $P(a) = 0$ .

**Definition 2.** A number  $a \in \mathbb{R}$  is called a **multiple root** of polynomial  $P(x)$  of multiplicity  $m$  if  $P(x)$  is divisible (without remainder) by  $(x - a)^m$  and not divisible by  $(x - a)^{m+1}$ .

If  $x_1$  is the root of a polynomial  $P_n(x)$  of degree  $n$ , then  $r = 0$ , and

$$P_n(x) = (x - x_1)Q_{n-1}(x), \quad (3)$$

where  $Q_{n-1}(x)$  is a polynomial of degree  $n - 1$ .  $Q_{n-1}(x)$  is simply the quotient, which can be obtained using the **polynomial long division**. Since  $x_1$  is known to be the root of  $P_n(x)$ , it follows that the remainder  $r$  must be zero.

If we know  $m$  roots,  $\{x_1, x_2, \dots, x_m\}$ , of a polynomial  $P_n(x)$  (why is it obvious that  $m \leq n$ ?), then, applying the above reasoning recursively,

$$P_n(x) = (x - x_1)(x - x_2) \dots (x - x_m)Q_{n-m}(x), \quad (4)$$

So, if we know that  $P_n(x)$  given by (1) has  $n$  roots,  $\{x_1, x_2, \dots, x_n\}$ , then,

$$P_n(x) = a_n(x - x_1)(x - x_2) \dots (x - x_n). \quad (5)$$

If two polynomials,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0$$

and

$$Q_n(x) = b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_2 x^2 + b_1 x^1 + b_0$$

are equal,  $P_n(x) = Q_n(x)$ , then all corresponding coefficients are equal,

$$a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} = b_{n-2}, \dots, a_{n-m} = b_{n-m}, \dots, a_1 = b_1, a_0 = b_0. \quad (6)$$

This is the easiest way to obtain Vieta's theorem and its generalizations for higher-order polynomials.

## Vieta theorem.

**Theorem.** Let  $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x^0 + a_0$  be a polynomial with leading coefficient 1 and roots  $x_1, x_2, \dots, x_n$ ,

$$f(x) = (x - x_1)(x - x_2) \dots (x - x_n).$$

Then the coefficients of  $f(x)$  can be written in terms of roots,

$$a_0 = (-1)^n x_1 x_2 \dots x_n$$

$$a_1 = (-1)^{n-1} (x_1 x_2 \dots x_{n-1} + x_1 x_2 \dots x_{n-2} x_n + \dots + x_2 x_3 \dots x_n)$$

...

$$a_{n-1} = -(x_1 + x_2 + \dots + x_n)$$

For  $n = 2$ , quadratic equation,  $x^2 + px + q = (x - x_1)(x - x_2)$ , we have,

$$q = x_1 x_2 \text{ and } p = -(x_1 + x_2)$$

For the cubic equation,  $n = 3$ , where  $x_1, x_2$  and  $x_3$  are the roots,

$$x^3 + a_2 x^2 + a_1 x + a_0 = (x - x_1)(x - x_2)(x - x_3),$$

$$a_0 = -x_1 x_2 x_3, a_1 = x_1 x_2 + x_2 x_3 + x_1 x_3, a_2 = -(x_1 + x_2 + x_3)$$

Moreover, any expression in the roots  $x_1, x_2, \dots, x_n$  which is symmetric (i.e., doesn't change when we permute any two roots) can be written in terms of the coefficients  $a_0, a_1, \dots, a_n$ . Example: for  $n = 2$ ,  $x_1^2 + x_2^2 = \dots$