

January 19, 2026

## Algebra.

### Maps. Functions. Injections, surjections, bijections.

A **map** is a rule that associates unique objects to elements in a given set. A **function** is a map that uniquely associates to **every** element of one set some element of another set:  $\forall a \in A, a \xrightarrow{f} f(a) = b \in B$ . A **partial function** on set  $A$  maps a subset of elements from set  $A$  on elements from set  $B$ .

**Definition.** A function is a relation that uniquely associates every member  $a$  in set  $A$  with some member  $b$  in set  $B$ , i.e. a function  $f$  is a map  $A \xrightarrow{f} B$  such that  $\forall a \in A, \exists! b \in B, b = f(a)$ .

Note that we do not require that every element  $b \in B$  appears as a value of a function. A function therefore can be one-to-one or many-to-one relation.

**Definition.** The set  $A$  of values at which a function  $f$  is defined is called its **domain**, while the set  $f(A)$  of values that the function can produce, which is a subset of  $B$ ,  $f(A) \subseteq B$ , is called its **range**. The set  $B$  is called the **codomain** of  $f$ .

**Definition.** For a subset  $X$  of the domain  $A$  of function  $f$ ,  $X \subseteq A$ , the **image**,  $f(X)$ , is the set of values  $y \in B$ ,  $y = f(x)$ ,  $\forall x \in X$ ,

$$Y = f(X) = \{y: (y \in B) \wedge (\exists x \in X, y = f(x))\}$$

**Definition.** For a subset  $Y$  of the range  $B$  of function  $f$ ,  $Y \subseteq B$ , the **pre-image**,  $f^{-1}(Y)$ , is the set of values  $x \in A$ ,  $y = f(x)$ ,  $y \in Y$ ,

$$X = f^{-1}(Y) = \{x: (x \in A) \wedge (\exists y \in Y, f(x) = y)\}$$

In particular, if an image  $Y = \{y\}$  is a single point  $y \in B$ , in this case,  $f^{-1}(\{y\})$  is the set of all solutions of the equation,  $f(x) = y$ .

**Exercise 1.** Let function  $f$  map  $A \xrightarrow{f} B$ . Prove that for any two subsets of its domain,  $X_1 \subset A, X_2 \subset A$ ,  $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$ .

Show that it could happen that  $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$  (hint: take  $X_1, X_2$  so that they do not intersect).

**Exercise 2.** Let function  $f$  map  $A \xrightarrow{f} B$ . Prove that for any two subsets of its co-domain,  $Y_1 \subset B, Y_2 \subset B, f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$ .

**Definition.** The function  $A \xrightarrow{f} B$  is **injective** (one-to-one) if every element of the co-domain  $B$  is mapped to by at most one element of the domain  $A$  (has no more than one pre-image),

$$\forall (x_1, x_2) \in A, (f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2), \text{ or,}$$

$$\forall (x_1, x_2) \in A, (x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))$$

An injective function is an **injection**.

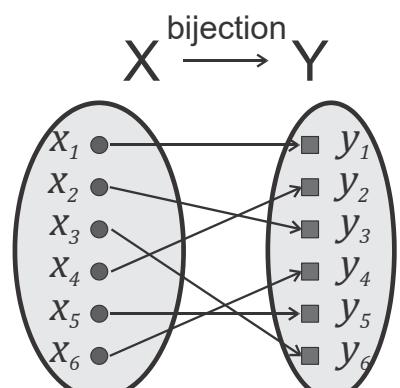
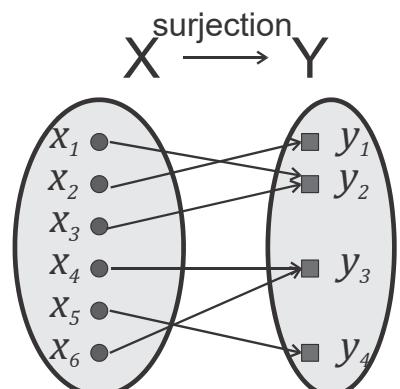
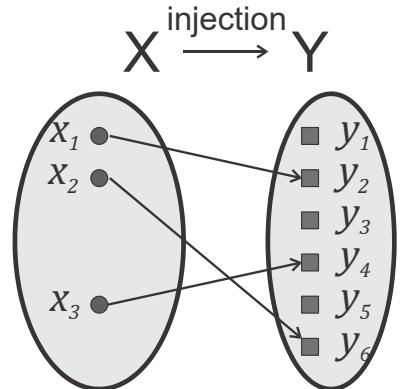
**Definition.** The function  $A \xrightarrow{f} B$  is **surjective** (onto) if every element of the co-domain  $B$  is mapped to by at least one element of the domain  $A$  (has pre-image in  $A$ ),

$$\forall y \in B, \exists x \in A: y = f(x).$$

That is, the image of the range of the surjective function coincides with the co-domain. A surjective function is a **surjection**.

An injective function need not be surjective (not all elements of the co-domain may have pre-images), and a surjective function need not be injective (some images may be associated with more than one pre-image).

**Definition.** The function  $A \xrightarrow{f} B$  is **bijective** (one-to-one correspondence, or one-to-one and onto) if every element of the co-domain is mapped to by exactly one element of the domain. That is, the function is both injective and surjective. A bijective function is a **bijection**.



A function is bijective if and only if every possible image is mapped to by exactly one argument (pre-image),

$$\forall y \in B, \exists! x \in A, y = f(x).$$

A function  $A \xrightarrow{f} B$  is bijective if and only if it is invertible, that is, there exists a function  $g, B \xrightarrow{g} A$  such that  $\forall x \in A, g(f(x)) = x$ , and  $\forall y \in B, f(g(y)) = y$ . Such a function is called inverse of  $f$  and denoted  $g = f^{-1}$ . This function maps each pre-image to its unique image. In other words,  $g \circ f = g(f(x))$  is an identity function on  $A$ , and  $f \circ g = f(g(y))$  is an identity function on  $B$ .

Bijections provide a way of comparing and identifying different sets. In particular, if there exists a bijection  $f$  between two finite sets  $A$  and  $B$ , then  $|A| = |B|$ .

**Exercise 1.** Show that  $f: A \xrightarrow{f} B$  is not injective exactly when one can find  $x_1, x_2 \in A$  such that  $x_1 \neq x_2$ , but  $f(x_1) = f(x_2)$ .

**Exercise 2.** Let  $f: A \xrightarrow{f} B$  and  $g: B \xrightarrow{f} C$  be bijections. Prove that the composition  $g \circ f: A \xrightarrow{g \circ f} C$ , defined by  $g \circ f(x) = g(f(x))$ , is also a bijection, and that so is  $(g \circ f)^{-1} = (f)^{-1} \circ (g)^{-1}$ .

**Exercise 3.** Construct bijections between the following sets:

1. (subsets of the set  $\{1, \dots, n\}$ )  $\leftrightarrow$  (sequences of zeros and ones of length  $n$ )
2. (5-element subsets of  $\{1, \dots, 15\}$ )  $\leftrightarrow$  (10-element subsets of  $\{1, \dots, 15\}$ )
3. [set of all ways to put 10 books on two shelves (order on each shelf matters)]  $\leftrightarrow$  (set of all ways of writing numbers  $1, 2, \dots, 11$  in some order)  
[Hint: use numbers  $1 \dots 10$  for books and 11 to indicate where one shelf ends and the other begins.]
4. (all integer numbers)  $\leftrightarrow$  (all even integer numbers)
5. (all positive integer numbers)  $\leftrightarrow$  (all integer numbers)
6. (interval  $(0,1)$ )  $\leftrightarrow$  (interval  $(0,5)$ )
7. (interval  $(0,1)$ )  $\leftrightarrow$  (halffline  $(1, \infty)$ ) [Hint: try  $1/x$ .]
8. (interval  $(0,1)$ )  $\leftrightarrow$  (halffline  $(0, \infty)$ )

9. (all positive integer numbers)  $\leftrightarrow$  (all integer numbers)

**Exercise 4.** Let  $A$  be a finite set, with 10 elements. How many bijections  $f : A \rightarrow A$  are there? What if  $A$  has  $n$  elements?

**Exercise 5.** Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be given by  $f(n) = 2n$ . Is this function injective? surjective?