MATH 9B: HOMEWORK 21 [MAY 3, 2026]

1. Algebra

1. **(i)** Find the magnitude and the argument, **(ii)** compute the inverse, and **(iii)** find the magnitude and the argument of the inverse for the following complex numbers:

(a) $1 + i$

(b) $-i$

(c) $1 + ix$

(d) $\frac{\sqrt{3}}{2} + \frac{i}{2}$

(e) $\frac{1}{2-i} - \frac{1}{2+i}$

2. Find a complex number z whose magnitude is 2 and whose argument is

$$\mathrm{Arg}(z) = \frac{\pi}{4} = 45^\circ.$$

3. Draw the following sets of points on the complex plane:

(a) $\{z \mid \mathrm{Re}(z) = 1\}$

(b) $\{z \mid \mathrm{Arg}(z) = \frac{3\pi}{4} = 135^\circ\}$

(c) $\{z \mid |z| = 1\}$

(d) $\{z \mid \mathrm{Re}(z^2) = 0\}$

(e) $\{z \mid |z^2| = 2\}$

(f) $\{z \mid |z-1| = 1\}$

(g) $\{z \mid z + \bar{z} = 1\}$

4. Prove that for any complex number z, we have:

(a) $|\bar{z}| = |z|, \quad \mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$

(b) $\frac{\bar{z}}{z}$ has magnitude 1; check this for $z = 1 - i$.

5. If z has magnitude 2 and argument $\frac{\pi}{2}$ and w has magnitude 3 and argument $\frac{\pi}{3}$, find the magnitude and argument of zw. Write the result in the form $a + bi$.

6. Let $P(x)$ be a polynomial with real coefficients.

(a) Prove that for any complex number z, we have $\overline{P(z)} = P(\bar{z})$.

(b) Let z be a complex root of this polynomial, $P(z) = 0$. Prove that then $\bar{z}$ is also a root, $P(\bar{z}) = 0$.

7. Solve the equation $x^3 - 4x^2 + 6x - 4 = 0$. Find the sum and the product of the roots in two ways: by using the Vieta formulas and by explicit computation. Check that the results match.

***8.** Let $P(x)$ be a polynomial with real coefficients, and with odd degree. Show that it has a real root.

2. Trigonometry

1. For an angle x which is not am integer multiple of π, show that

$$\cos(x)\cos(2x)\cos(4x)\ldots\cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin(x)}$$

2. Consider a circle of unit radius as in the figure, and let AB be a chord subtending an angle x at the center O. By comparing the areas of the triangle OAB, the sector OAB, and the triangle OBC, show that the following inequalities are true:

$$1 > \frac{\sin x}{x} > \cos x$$

Therefore, show that $\sin(x)/x$ approaches 1 as x approaches 0.

3. Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

(Hint: use $x = \pi/2^{n+1}$ in Problem 1, and then let n go to infinity. Use problem 2, as well as the expressions for $\cos(\pi/4), \cos(\pi/8), \ldots$ that we calculated using the half angle formulae.)