Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homeworks. Try solving the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a\in\mathbb{Q}_<,\ \forall b\in\mathbb{Q}_>,\ a < b$.
   a. Prove that if $\mathbb{Q}_>$ contains a smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b\in\mathbb{Q}_>, b_0\leq b$, then $\mathbb{Q}_<$ does not contain a largest element.
   b. Prove that if $\mathbb{Q}_<$ contains a largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a\in\mathbb{Q}_<, a\leq a_0$, then $\mathbb{Q}_>$ does not contain a smallest element.
   c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains a smallest element, nor $\mathbb{Q}_<$ contains a largest element.
2. Prove the following properties of countable sets.
   a. For any two countable sets, A, B, their union, $A\cup B$ is also countable, $\left(\left(c(A) = \aleph_0\right) \wedge \left(c(B) = \aleph_0\right)\right) \Rightarrow \left(c(A\cup B) = \aleph_0\right)$
   b. For any two countable sets, A, B, their Cartesian product, $A\times B = \{(a, b), a\in A,\ b\in B\}$, is also countable, $\left(\left(c(A) = \aleph_0\right) \wedge \left(c(B) = \aleph_0\right)\right) \Rightarrow \left(c(A\cup B) = \aleph_0\right)$
   c. For a finite collection of countable sets, $\{A_1,\ A_2,\ ...,\ A_n\}$, $c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 ... \cup A_n) = \aleph_0$
   d. For a countable collection of countable sets, $\{A_1,\ A_2,\ ...,\ A_n,\ ...\ \}$, $c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 ... \cup A_n \cup ...) = \aleph_0$
3. Let W be the set of all "words" that can be written using the alphabet consisting of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]

4. Compare the following real numbers (are they equal? which is larger?)
   a. $1.33333... = 1.(3)$ and $4/3$
   b. $0.09999... = 0.0(9)$ and $1/10$
   c. $99.9999... = 99.(9)$ and 100
   d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Simplify the following real numbers. Are these numbers rational? (hint: you may use the formula for an infinite geometric series).
   a. $1/1.1111...=1/1.1(1)$
   b. $2/1.2323...=2/1.23(23)$
   c. $3/0.123123...=3/0.123(123)$
6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. $1/8$
   b. $2/7$
   c. 0.1
   d. $0.33333... = 0.(3)$
   e. $0.13333... = 0.1(3)$
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\, a, b \in R$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in R, \exists c \in R, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in R, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in R,\ a > b > 0,\ \exists n \in N$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in R, a + b = b + a$
- $\forall\, a, b, c \in R, (a + b) + c = a + (b + c)$
- $\forall\, a \in R, \exists 0 \in R, a + 0 = a$
- $\forall\, a \in R,\ \exists - a \in R, a + (- a) = 0$
- $\forall\, a, b \in R, a - b = a + (- b)$
- $\forall\, a, b, c \in R, (a < b) \Rightarrow (a + c < b + c)$.