MATH 9B: HOMEWORK 13 (ASSIGNED JAN 25; DUE FEB 1)

1. Algebra

1. Construct bijections between the following sets:
  (a) (subsets of the set $\{1,\dots,n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n).
  (b) (5-element subsets of $\{1,\dots,15\}$) $\leftrightarrow$ (10-element subsets of $\{1,\dots,15\}$).
  (c) (set of all ways to put 10 books on two shelves (order on each shelf matters)) $\leftrightarrow$ (set of all ways of writing numbers $1,2,\dots,11$ in some order) [Hint: use numbers $1,\dots,10$ for books and 11 to indicate where one shelf ends and the other begins.]
  (d) (all integer numbers) $\leftrightarrow$ (all even integer numbers)
  (e) (all positive integer numbers) $\leftrightarrow$ (all integer numbers)
  (f) (interval (0,1)) $\leftrightarrow$ (interval (0,5))
  (g) (interval (0,1)) $\leftrightarrow$ (halfline (1, ∞)) [Hint: try $1/x$.]
  (h) (interval (0,1)) $\leftrightarrow$ (halfline (0, ∞))
  (i) * (all positive integer numbers) $\leftrightarrow$ (all positive rational numbers)

2. Let A be a finite set, with 10 elements. How many bijections $f : A \to A$ are there? What if A has n elements?

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = 2n$. Is this function injective? surjective?

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2$. Is this function injective? surjective?

5. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered $1,2,3,\dots$. Each hotel room is single occupancy: only one guest can stay there at any time.
  (a) At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how? (Hint: Construct a bijection between the set $\{-1,0,1,2,\dots,\}$ and the set of natural numbers $\mathbb{N}$).
  (b) At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2,4,6,\dots\}$ and $\mathbb{N}$).
  (c) Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms numbered by all integers: $\{\dots,-2,-1,0,1,2,\dots\}$. Yet, the management of the original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{\dots,-2,-1,0,1,2,\dots\}$ and $\mathbb{N}$).

2. Bonus algebra review problems

1. Using the recurrence relation $d_n = (n-1)(d_{n-1} + d_{n-2})$ obtained by solving the old hats problem, derive the formula for a derangement probability,

$$p_n = \frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

2. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are divisible by 5, 7, 11, or 13.

3. How many passwords of at least 8 characters can one compose using lower- and upper-case letters and numbers 0 to 9?

4. * If 9 dies are rolled, what is the probability that all 6 numbers appear?

5. * How many permutations of the 26 letters of English alphabet do not contain any of the words pin, fork, or rope?

3. Geometry

1. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).

2. (Skanavi 15.104) Given the right triangle ABC with $\angle C = 90°$, find the locus of points $P(x, y)$ such that $|PA|^2 + |PB|^2 = 2|PC|^2$.

3. (Skanavi 15.109) Points $A(-1, 2)$ and $B(4, -2)$ are vertices of the rhombus $ABCD$, while point $M(-2, 0)$ belongs to the side CD. Find the coordinates of the vertices C and D.

4. (Skanavi 15.114) Find the circle (write the equation of this circle) passing through the coordinate origin $O(0, 0)$, point $A(1, 0)$, and tangent to the circle $x^2 + y^2 = 9$.

5. (Skanavi 15.115) Write the equation of the circle passing through the point $A(2, 1)$ and tangent to both X- and Y-axes.

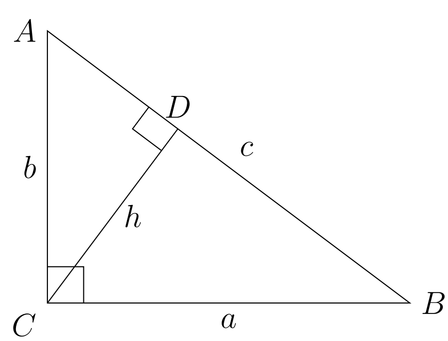
6. In the notation of the classwork,

(a) Find the orthocenter of H the triangle $\triangle ABC$. [Hint: calculate the equations for two of the altitudes]

(b) Show that H, O, G are on a straight line, and that $HG = 2GO$.

7. (a) In the figure below, find h in terms of a and b by applying Pythagoras's theorem and calculating the area of the right triangle in two different ways.

(b) Find the distance of the origin $(0, 0)$ from the line $ax + by = c$. [Hint: you can assume a, b, c are positive, and draw an appropriate right triangle as in the previous part].



8. Calculate the area of the triangle with vertices $A = (0, 0)$, $B = (x_1, y_1)$ and $C = (x_2, y_2)$. [Hint: use the formula (1/2) base · height, where the base is BC, and the height follows from the previous problem once you write down the equation of the line BC.]