Homework for January 11, 2026 (due Jan 18).

Algebra.

Review the last classwork handout. Review and solve the classwork exercises which were not solved and unsolved problems from the previous homeworks. Solve the following problems (skip the ones that you have already solved).

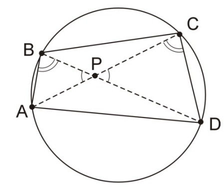
1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}, B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}, B = \{1,15\}$
   c. $A = \varnothing, B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1], B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1], B = (-\infty, \infty)$
   c. $A = (-\infty, 0], B = [0, \infty)$
   d. $A = (-\infty, \infty), B = (-\infty, \infty)$
   e. $A = [0,1), B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.
   a. What is the number of elements in a set $A \times A$
   b. What is the number of elements in a set $A \times (A \times A)$

Geometry.

Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proofs of Ptolemy's and Euclid's theorems. Try solving the following problems including the unsolved problems from previous homeworks (problems marked with asterisk are optional – you are not expected to solve them).

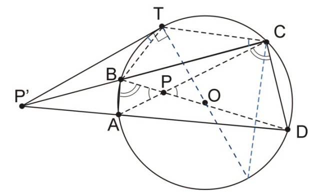
6. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.

a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?

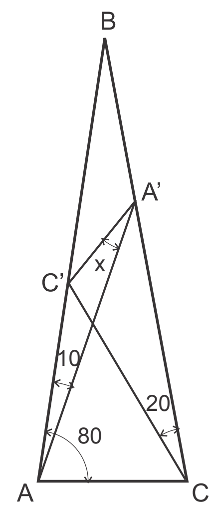


b. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?

7. Using the Ptolemy's theorem, prove the following:



   a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
   c. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle?
8. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
9. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.



10. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
11. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.
12. * Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$\left|AC\right| \bullet \left|BD\right| \leq \left|AB\right| \bullet \left|CD\right| + |BC| \bullet |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).