MATH 9B: HOMEWORK 20 [APRIL 26, 2026]

ALGEBRA

(1) Let x_1, x_2, x_3 be distinct real numbers. Prove that there exists a unique polynomial $P(x)$ of degree 2 such that $P(x_1) = 1, P(x_2) = P(x_3) = 0$.
[Hint: if $P(x_1) = 0$, then $P(x)$ is divisible by $(x - x_1)$.]
Find this polynomial if $x_1 = 2$, $x_2 = -1$, $x_3 = 5$.

(2) Let x_1, x_2, x_3 be distinct real numbers and y_1, y_2, y_3 any numbers. Prove there exists a unique quadratic polynomial $f(x)$ such that

$$f(x_i) = y_i, \quad i = 1, 2, 3.$$

Find it for $x_1 = 2$, $x_2 = -1$, $x_3 = 5$, $y_1 = 3$, $y_2 = 6$, $y_3 = 18$.
[Hint: look for it in the form $f(x) = y_1 f_1(x) + \dots$].

(3) Prove that given distinct $x_1, \dots, x_n$ and arbitrary $y_1, \dots, y_n$, there exists a unique polynomial of degree $n - 1$ such that

$$f(x_i) = y_i, \text{ for } i = 1, \dots, n$$

(For $n = 2$, this is a statement that there is a unique line through two given points.)

(4) Prove that if $P(x)$ has integer coefficients, then for any integers a, b,

$$P(a) - P(b)$$

is divisible by $a - b$.

(5) Let x_1, x_2 be the roots of $x^2 + 7x - 3 = 0$. Find:
(a) $x_1^2 + x_2^2$
(b) $\dfrac{1}{x_1} + \dfrac{1}{x_2}$
(c) $(x_1 - x_2)^2$
(d) $x_1^3 + x_2^3$

(6) What is the maximum number of integer roots that the following polynomial can have?

$$x^{10} + ax^9 + bx^8 + cx^7 + dx^6 + fx^5 + gx^4 + hx^3 + kx^2 + lx + m = 1024$$

How does the answer change if $a = 0$?

(7) Compute
(a) $(1 + i)^{-1}$
(b) $\dfrac{1 + i}{1 - i}$
(c) $(1 + i)^{-3}$

(8) Solve the following equations in complex numbers:
(a) $z^2 = -2 + 2i\sqrt{3}$
(b) $z^2 - z + 1 = 0$.

(1) Find all x such that:

(a) $\sin x \cos x = \frac{\sqrt{2}}{2}$

(b) $\sin x \cos x = \frac{1}{2}$

(c) $\sin x \cos x = \frac{\sqrt{3}}{4}$

(2) Find the sum

$$S = \cos x + \cos 3x + \cos 5x + \cdots + \cos 2025x.$$

(Hint: multiply the sum by $2\sin x$)

(3) Calculate:

(a) $\cos 75^\circ + \cos 15^\circ$

(b) $\cos \frac{\pi}{12} - \cos \frac{5\pi}{12}$

(4) Let A, B, C be angles of a triangle. Prove:

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

(5) Prove:

(a) $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$

(b) $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$

(c) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$

(d) $\frac{\cot^2 2\alpha - 1}{2\cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$

(e) $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$

(f) $\frac{\sin 6\alpha + \sin 7\alpha + \sin 8\alpha + \sin 9\alpha}{\cos 6\alpha + \cos 7\alpha + \cos 8\alpha + \cos 9\alpha} = \tan\frac{15\alpha}{2}$

(g) $\sin^6\alpha + \cos^6\alpha = \frac{5 + 3\cos 4\alpha}{8}$

(h) $16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha = \sin 5\alpha$

(i) $\frac{\cos 64^\circ \cos 4^\circ - \cos 86^\circ \cos 26^\circ}{\cos 71^\circ \cos 41^\circ - \cos 49^\circ \cos 19^\circ} = -1.$

(j) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

(k) $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$

(6) Simplify:

(a) $\sin^2\left(\frac{\alpha}{2}+2\beta\right)-\sin^2\left(\frac{\alpha}{2}-2\beta\right)$

(b) $2\cos^2 3\alpha+\sqrt{3}\sin 6\alpha-1$

(c) $\cos^4 2\alpha-6\cos^2 2\alpha\sin^2 2\alpha+\sin^4 2\alpha$

(d) $\sin^2(135^\circ-2\alpha)-\sin^2(210^\circ-2\alpha)-\sin^2 195^\circ\cos(165^\circ-4\alpha)$

(e) $\dfrac{\cos 2\alpha-\cos 6\alpha+\cos 10\alpha-\cos 14\alpha}{\sin 2\alpha+\sin 6\alpha+\sin 10\alpha+\sin 14\alpha}$

(7) Solve:

(a) $\cos^2(\pi x)+4\sin(\pi x)+4=0$

(b) $\sin x+\sin 2x+\sin 3x=\cos x+\cos 2x+\cos 3x$

(c) $\cos 3x-\sin x=\sqrt{3}(\cos x-\sin 3x)$

(d) $\sin^2 x-2\sin x\cos x=3\cos^2 x$

(e) $\sin 6x+2=2\cos 4x$

(f) $\cot x-\tan x=\sin x+\cos x$

(g) $\sin x\geq\frac{\pi}{2}$

(h) $\sin x\leq\cos x$