Homework 19 for April 12, 2025.

Algebra.

Read the classwork handout.

1. Perform long division of the following polynomials.
   a. $\left(x^5 - 2x^3 + 3x^2 - 4\right) \div \left(x^2 - x + 1\right)$
   b. $\left(x^4 - x^2 + 1\right) \div (x + 1)$
   c. $\left(x^7 + 1\right) \div \left(x^3 - x + 1\right)$
   d. $\left(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1\right) \div \left(x^2 + 1\right)$
   e. $\left(x^5 - 32\right) \div (x + 2)$
   f. $\left(x^5 - 32\right) \div (x - 2)$
   g. $\left(x^6 + 64\right) \div \left(x^2 + 4\right)$
   h. $\left(x^6 + 64\right) \div \left(x^2 - 4\right)$
   i. $\left(x^{100} - 1\right) \div \left(x^2 - 1\right)$
2. Can you find coefficients a, b, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,
   a. by $x + 5$
   b. by $x^2 + x - 1$
3. Prove that,
   a. for odd n, the polynomial $x^n + 1$ is divisible by $x + 1$
   b. $2^{100} + 1$ is divisible by 17.
   c. $2^n + 1$ can only be prime if n is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
   d. for any natural number n, $8^n - 1$ is divisible by 7.
   e. for any natural number n, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

   a. $1 + a + a^2 + a^3$
   b. $1 - a + a^2 - a^3 + a^4 - a^5$
   c. $a^3 + 3a^2b + 3b^2a + b^3$
   d. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.
   e. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$
   f. $\frac{x^6-1}{x^4+x^2+1}$
   g. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$
6. Solve the following equations
   h. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)
   i. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)
7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of p and q,
   j. $\left(x_1 + x_2 + x_3\right)^2$
   k. $x_1^2 + x_2^2 + x_3^2$
   l. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   m. $\left(x_1 + x_2 + x_3\right)^3$
8. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

   Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it:
   $x^4 - x^3 + 5x^2 - x - 6.$

Geometry.

Read the classwork handout. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163), http://en.wikipedia.org/wiki/Trigonometric_functions http://en.wikipedia.org/wiki/Sine. Solve the following problems.

1. Derive the following expressions (classwork exercise):
   a. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha)\cos(\beta)}$ $\quad\cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha\pm\beta)}{\sin(\alpha)\sin(\beta)}$

   b. $\sin^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$

   c. $\cos^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$

   d. $= \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

   e. $\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 - \cos\alpha)}$ $\quad\cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 + \cos\alpha)}$

   f. $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$

   g. $\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$

   h. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

   i. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

   j. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

2. Show that

   a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3} + \alpha\right) + \cos^2\left(\frac{2\pi}{3} - \alpha\right) = \frac{3}{2}$

   b. $\sin\alpha + \sin\left(\frac{2\pi}{3} + \alpha\right) + \sin\left(\frac{4\pi}{3} + \alpha\right) = 0$

   c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

3. Without using calculator, find:

   a. $\sin 75° =$

b. $\cos 75° =$

c. $\sin \frac{\pi}{8} \quad =$

d. $\cos \frac{\pi}{8} \quad =$

e. $\sin \frac{\pi}{16} \quad =$

f. $\cos \frac{\pi}{16} \quad =$