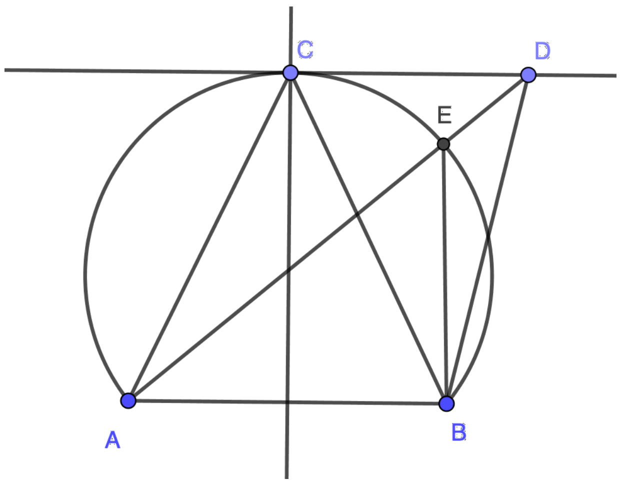
February 22, 2026 Math 9b

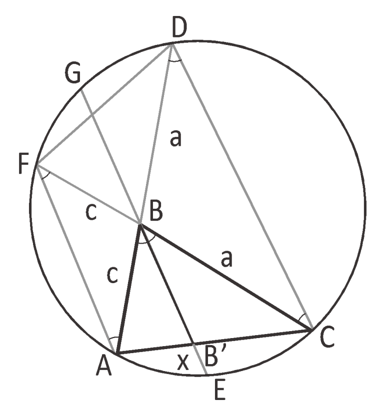
Solutions of some past homework problems.

1. **Problem**. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.



Solution. Consider isosceles triangle ABC with AC = BC. Consider the circumcircle of triangle ABC and its tangent line at C. This tangent line is parallel to AB, and so any triangle ABD with the same base AB and the same height as ABC must have D on the tangent line. Now, we have $\angle ACB = \angle AEC = \angle ADB + \angle EBD > \angle ADB$.

2. **Problem**. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.



Solution. Consider the construction used to prove the property of a bisector: an isosceles triangle CBD obtained by extending the side AB and marking off a point D, such that $CB = BD = a$. (Recap: the property of a bisector, BB', is obtained by applying Thales theorem to the angle DAC and two parallel lines, BB' and CD; we then obtain, $|AB'|:|B'C| = |AB|:|BC|$).

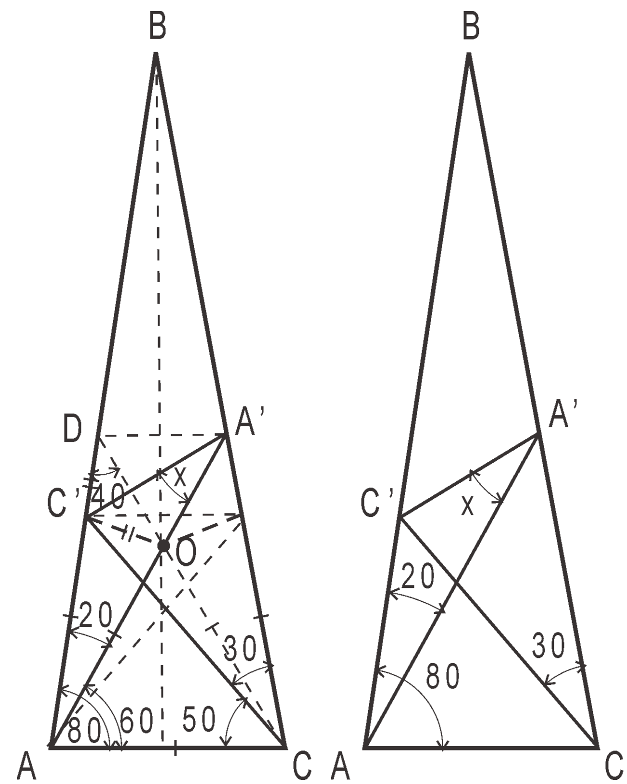
Similarly, extend side CB and mark off a point F such that $FB = AB = c$. Then $BB' \parallel CD$ and also $BB' \parallel AF$. In particular, $\angle AFC = \angle ADC = \angle ABC/2$.

Therefore $AFDC$ is a cyclic quadrilateral.

Draw the circumscribed circle around $AFDC$ and extend the bisector BB to obtain the chord EG containing BB'. By symmetry, $|EB| = |BG|$ (see Figure). By the property of intersecting chords (Euclid’s theorem), we have,

$|AB||BD| = |EB||BG| = |EB|^2 = \left(|BB'| + |B'E|\right)^2$, wherefrom,
$|BB'|^2 = |AB||BD| - |B'E|\left(|B'E| + 2|BB'|\right)$. On the other hand, by the same theorem,
$|B'E||B'G| = |B'E|\left(|B'E| + 2|BB'|\right) = |AB'||B'C|$. Combining these two expressions, we obtain $|BB'|^2 = |AB||BC| - |AB'||B'C|$.

3. **Problem**. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



Solution. Consider the figure. (reflect A across the axis of symmetry to get D on AB), and let AA’ and CD intersect in O, which is on the axis of symmetry.

We see that triangle $\Delta A'CC'$ is isosceles, so $|AC'| = |AC|$.

Next,
$\angle C'DO = \angle OA'C = \angle A'BA + \angle A'AB = 20° + 20° = 40°$,

so $\angle AOC = \angle ODA + \angle OAD = 40° + 20° = 60°$. It follows that ΔOAC, which is clearly isosceles, is in fact equilateral (and so is $\Delta A'OD$). Next, note that $|AC'| = |AC| = |AO|$, so triangle $\Delta AC'O$ is isosceles, and $\angle AC'O = \angle AOC' = 80°$.

Now, $\angle C'DO = 40°$ implies $\angle C'OD = \angle AC'O - \angle C'DO = 80° - 40° = 40°$. Therefore $|C'D| = |C'O|$ and so triangles $\Delta A'DC'$ and $\Delta A'OC'$ are congruent by SSS. So $\angle OA'C' = ½\,\angle OA'D = 30°$.