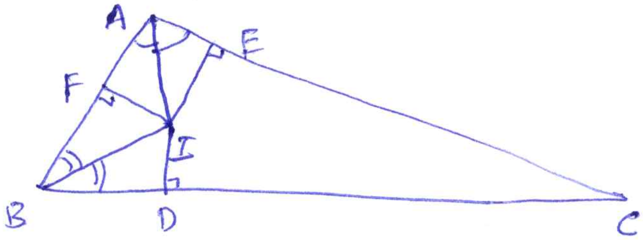
GEOMETRY

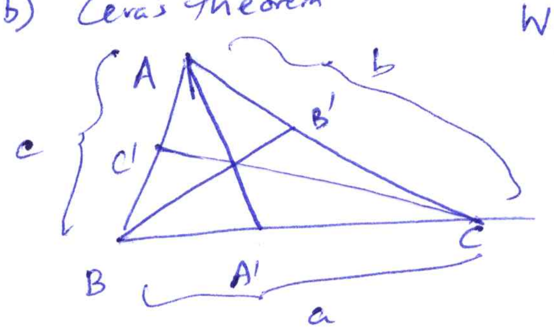
1. Angular bisectors of a Δle are concurrent – 3 proofs.

a) From first principles (last yr)



Let angular bisectors of A, B meet in I. Let perpendiculars from I to BC, AC, AB be D, E, F resp. Then $IE = IF$ (since I is on bisector of $\angle A$) and $IF = ID$ (since I is on bisector of $\angle B$) so $IE = ID \Rightarrow I$ is on bisector of $\angle C$.

b) Ceva's theorem



We showed $\frac{BA'}{A'C} = \frac{AB}{AC}$ by the property of angle bisector

Similarly $\frac{CB'}{B'A} = \frac{BC}{AB}$

$\frac{AC'}{C'B} = \frac{AC}{BC}$

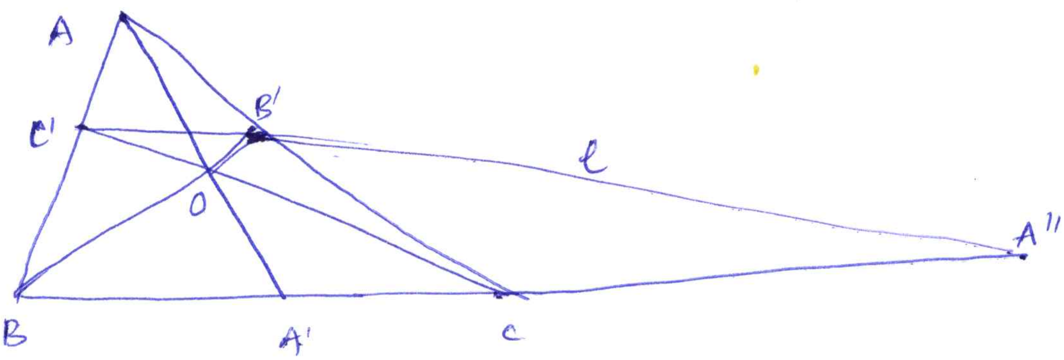
Multiply to get 1. So the 3 Cevians AA', BB', CC' are concurrent by Ceva's theorem converse.

c) Method of masses. We want to put masses m_A, m_B, m_C at A, B, C resp, such that if we combine m_B at B and m_C at C their COM is at A'. Need $\frac{m_B}{m_C} = \frac{CA'}{A'B} = \frac{AC}{AB} = \frac{b}{c}$.

That suggests we should put weight b at B, c at C, and a at A. Then COM lies on AA', and on BB', and on CC'. So these 3 are concurrent.

2. Relation between Ceva's and Menelaus's theorem

The products of ratios which show up in Ceva's and Menelaus's theorems are very similar. Consider the picture below



We start with a Δle ABC and 3 concurrent Cevians which meet at O, AA', BB', CC'. Now extend $C'B'$ (line l) to intersect BC at A'',

Ceva says: $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$

Menelaus says $\frac{AC'}{C'B} \cdot \frac{BA''}{A''C} \cdot \frac{CB'}{B'A} = 1$

so $\frac{BA'}{A'C} = \frac{BA''}{A''C}$. This is a property of the 4 points B, A', C, A'' on the line BC. The points B, C, A', A'' (in that order) are said to be in <u>harmonic range</u>. The relation is more properly written as $\frac{BA'/A'C}{BA''/A''C} = -1$ (the negative sign because $A''C$ is in the opposite direction as CA''). We will explore this property more in the homework. More generally, we can define a number corresponding to 4 points on a line, as follows.

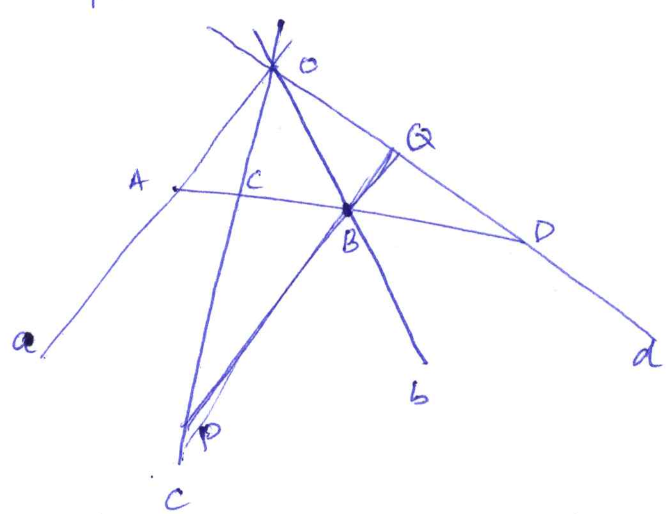
3. Cross - Ratio

The cross-ratio of the 4 ordered points A, B, C, D, written as $(A, B; C, D)$, is $\dfrac{AC/CB}{AD/DB}$

Note: If $m = (A, B; C, D)$ then $(B, A; C, D) = (A, B; D, C) = 1/m$ (see Homework)

Also $(A, C; B, D) = 1 - m$

Theorem (4 lines through a point) Let a, b, c, d be four lines through a point O, and let an arbitrary line ℓ, not through O, meet them in A, B, C, D respectively. The value of the cross-ratio $(A, B; C, D)$ is independent of the line ℓ. [It only depends on the position of the lines a, b, c, d].



Proof: Through B draw a line parallel to OA, meeting OC, OD in P, Q resp. By similar Δles, $\dfrac{AC}{CB} = \dfrac{OA}{BP}$, and $\dfrac{AD}{DB} = \dfrac{OA}{BQ}$

So $\dfrac{AC/CB}{AD/DB} = \dfrac{BQ}{BP}$. But note that this ratio is the same, whatever the point B is chosen on the line b (again by similar Δles) $\square$