

MATH 9B: TRIGONOMETRY [MARCH 22, 2026]
DEFINITION OF \sin , \cos , ETC. TRIANGLE FORMULAE.

TRIGONOMETRY: DEFINITION OF $\sin \alpha$ AND $\cos \alpha$

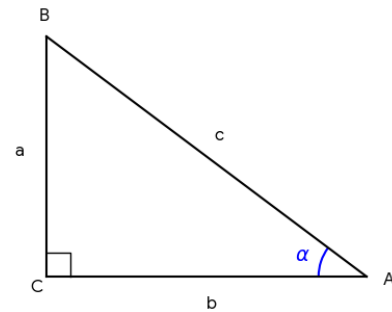
For any acute angle α , we can draw a right triangle that includes this angle α .

Definition: The sine of angle α is the ratio of the length of the leg opposite to this angle to the length of the hypotenuse of the triangle.

$$\sin \alpha = \frac{a}{c}$$

Definition: The cosine of α is the ratio of the length of the leg adjacent to this angle to the length of the hypotenuse of the triangle.

$$\cos \alpha = \frac{b}{c}$$



If α and β are complementary angles, then $\sin \alpha = \cos \beta$ and $\cos \alpha = \sin \beta$. If a and b are a pair of numbers such that $a^2 + b^2 = 1$, then there exists an angle α such that $a = \sin \alpha$ and $b = \cos \alpha$.

Definition: Tangent and cotangent of angle α are:

$$(1) \quad \tan \alpha = \frac{a}{b} = \frac{\sin \alpha}{\cos \alpha}$$

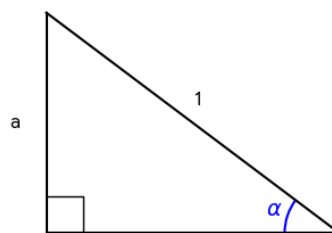
$$(2) \quad \cot \alpha = \frac{b}{a} = \frac{\cos \alpha}{\sin \alpha} = \frac{1}{\tan \alpha}$$

Theorem: The values of the trigonometric functions of an acute angle depend only on the size of the angle itself, and not on the particular right triangle containing the angle. This theorem follows immediately from the Thales theorem.

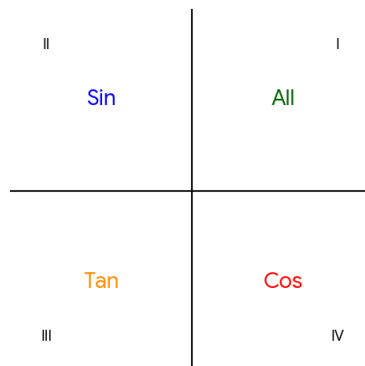
Exercises.

1. Show that $\sin^2 \alpha + \cos^2 \alpha = 1$.
2. Show that $\sin \alpha \leq 1$, $\cos \alpha \leq 1$, but $\sin \alpha + \cos \alpha \geq 1$.
3. Compute $\sin 45^\circ$, $\cos 45^\circ$, $\sin 30^\circ$, $\cos 30^\circ$, $\sin 60^\circ$, $\cos 60^\circ$.
4. Fill in the following table:

	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	$\cot \alpha$
$\sin \alpha$	a	$\sqrt{1 - a^2}$	$\frac{a}{\sqrt{1 - a^2}}$	$\frac{\sqrt{1 - a^2}}{a}$
$\cos \alpha$		a		
$\tan \alpha$			a	
$\cot \alpha$				a



Trigonometric Functions for angles outside $[0, \pi/2]$. We can work out formulas like $\sin(\pi - \theta) = \sin \theta$ and $\cos(\pi - \theta) = -\cos \theta$, using the angle-sum formulas to follow. Alternatively, we can use our intuition regarding when a length is “signed”. For now, note the signs of the trigonometric functions \sin , \cos , \tan in the four quadrants are as in the figure to the right. For example, “Tan” in the third quadrant means that \tan is positive, and \sin and \cos are negative there.



Inverse Trigonometric Functions. If we are given an angle, 30° for example, then we can find $\sin(30^\circ) = 1/2$. Inversely, if we are given a value of the sine function, say $1/2$, the challenge is to name the angle x such that $\sin x = 1/2$.

$$\arcsin \frac{1}{2} = \arcsin(\sin(30^\circ)) = 30^\circ$$

The Law of Sines. For a triangle ABC, we have

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

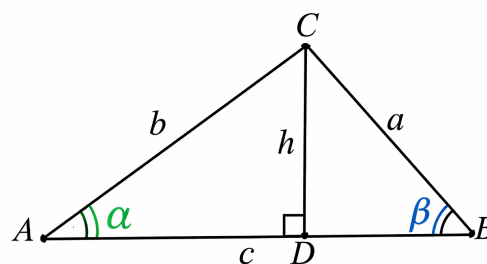
Proof. : Consider the figure to the right. We have

$$h = b \sin \alpha = a \sin \beta$$

from which

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

follows. Similarly the third equality follows by considering a different altitude. \square



The Law of Cosines. For any triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

Proof: Consider right triangles formed by height CD in the figure above:

$$a^2 = h^2 + |DB|^2$$

$$|DB| = c - |AD| = c - b \cos \alpha$$

$$h = b \sin \alpha$$

$$\begin{aligned} \text{So: } a^2 &= b^2 \sin^2 \alpha + (c - b \cos \alpha)^2 \\ &= c^2 + b^2(\sin^2 \alpha + \cos^2 \alpha) - 2bc \cos \alpha \\ &= b^2 + c^2 - 2bc \cos \alpha. \end{aligned}$$

Trigonometric Formulae. Addition Formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

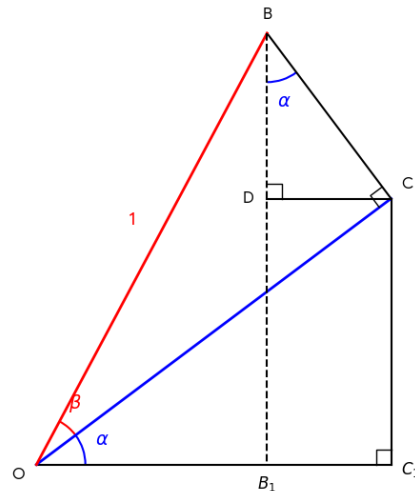
Proof. : Consider the figure to the right. We have $\sin(\alpha + \beta) = BB_1$, by definition. Now,

$$\begin{aligned} BB_1 &= BD + DB_1 \\ &= BC \cos \alpha + CC_1 \\ &= \sin \beta \cos \alpha + OC \sin \alpha \\ &= \sin \beta \cos \alpha + \cos \beta \sin \alpha \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta. \end{aligned}$$

Similarly,

$$\begin{aligned} \cos(\alpha + \beta) &= OB_1 = OC_1 - B_1C_1 \\ &= OC \cos \alpha - CD \\ &= \cos \beta \cos \alpha - BC \sin \alpha \\ &= \cos \alpha \cos \beta - \sin \beta \sin \alpha \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned}$$

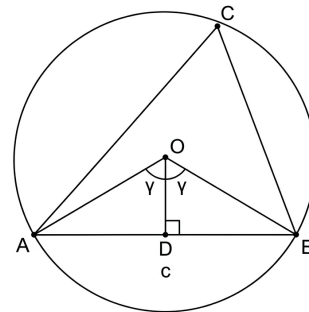
□



The Extended Law of Sines: For any triangle ABC with circumscribed circle radius R :

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

Proof. : Consider the figure to the right. We have $OA = OB = R$, and $AD = DB = c/2$. By definition $R \sin \gamma = c/2$, and therefore $c = 2R \sin \gamma$. Similarly for a and b . □



Area of a Triangle. Standard formulas for $S_{\Delta ABC}$: From the figure above we obtain

$$S_{\Delta ABC} = \frac{1}{2}hc = \frac{1}{2}bc \sin \alpha \left(= \frac{1}{2}ab \sin \gamma = \frac{1}{2}ca \sin \beta \right)$$

Using the extended Law of Sines:

$$S_{\Delta ABC} = \frac{abc}{4R} = 2R^2 \sin \alpha \sin \beta \sin \gamma$$

Using the inscribed radius r and semiperimeter s :

$$S_{\Delta ABC} = sr$$

Heron's Formula:

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

Exercise: Derive the addition formula for sine and cosine using the figure of the triangle with an altitude drawn (on page 2).

Exercise: Using the expression for the cosine of the sum of two angles derived above, derive the expression for the sine of the sum of two angles.