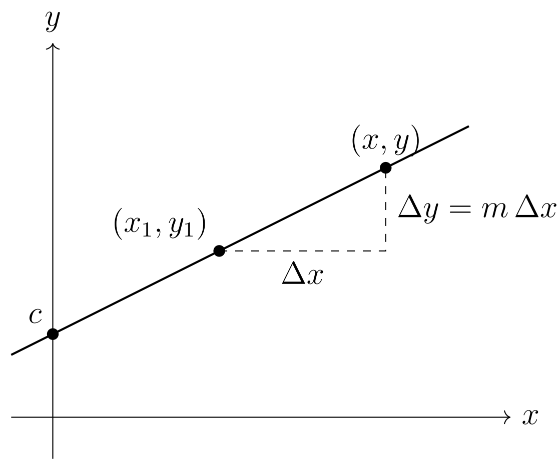
MATH 9B: GEOMETRY CLASSWORK [JAN 25, 2026]
COORDINATE GEOMETRY: LINES, TRIANGLES.

1. Straight lines



A line in the xy coordinate plane can usually be written in the form $y = mx + c$. Here m is the slope of the line: if we change x by 1 unit, the value of y will change by m units. The other parameter, c, is the intercept of the line on the y-axis, i.e. the value of y when $x = 0$.

Example: Suppose we want to write the equation of the line of slope m, passing through a point (x_1, y_1). We can write this equation as

$$\frac{y - y_1}{x - x_1} = m \Longrightarrow y = mx + (y_1 - mx_1).$$

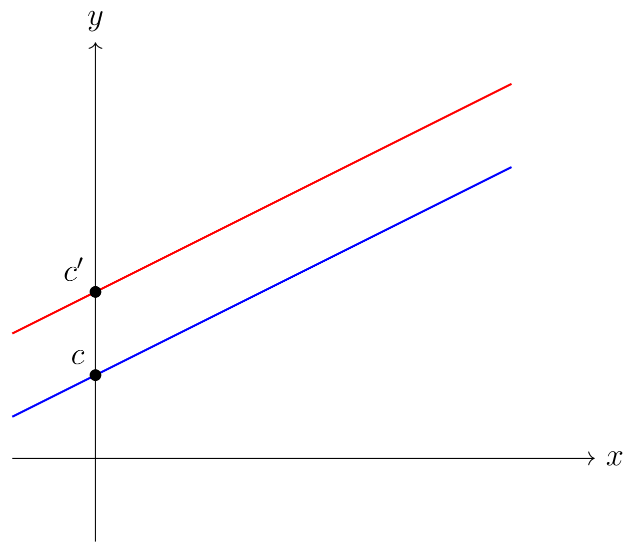
Example: What is the line joining the points (x_1, y_1) and (x_2, y_2)? Here the equation can be written as

$$\begin{aligned} & \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \\ \Longrightarrow \quad & y(x_2 - x_1) - x(y_2 - y_1) = y_1 x_2 - x_1 y_2 \\ \text{or} \quad & y = \frac{y_2 - y_1}{x_2 - x_1} + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}. \end{aligned}$$

Another way to obtain this equation is to use the previous formula; we know that a line going through (x_1, y_1) has the equation

$$y = mx + (y_1 - mx_1).$$

for some m. To figure out m, we can use the fact that the line goes through (x_2, y_2); namely, plug in $x = x_2$ and $y = y_2$ into the equation and solve for m.

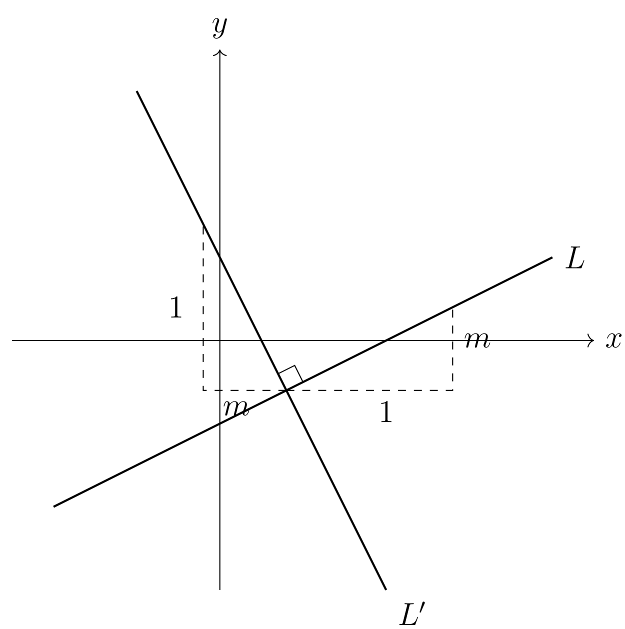


Parallel lines have the same slope, so we can write a pair of parallel lines as $y = mx + c$ and $y = mx + c'$.

What if the slope of a line is infinite? Then it is parallel to the y-axis, and of the form $x = c$ for some constant c. Note that there is no way to write this in the form $y = mx + c$. So the more general way to write a plane in the line is

$$ax + by = d.$$

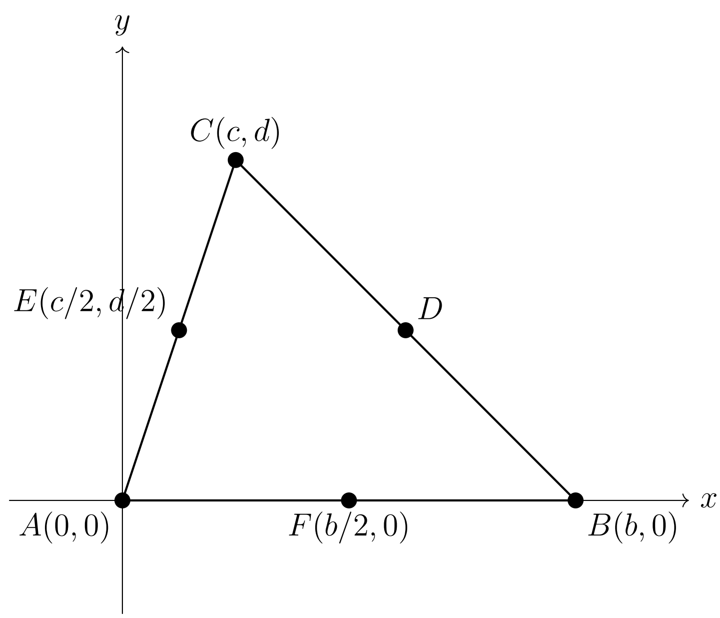
When $b \neq 0$, we can divide by b and rewrite in the form $y = mx + c$.



Perpendicular lines have slopes which multiply to -1 (in general, when both slopes are finite numbers).

Example: The lines perpendicular to $ax + by = c$ are of the form $bx - ay = d$, for all possible values of d.

2. TRIANGLES



We can put the three vertices of a $\triangle ABC$ at $A = (0, 0)$, $B = (b, 0)$, $C = (c, d)$, without losing any generality (why?)

Let's figure out the positions and equations of some standard points and lines corresponding to a triangle. The **midpoint** F of AB is at $(b/2, 0)$; we can think of this as $\frac{1}{2}(A+B)$, by identifying points with vectors. Similarly, the midpoint E of AC is at $(c/2, d/2)$, and the midpoint D of BC is at $((b + c)/2, d/2)$.

Next, let's calculate the equation of the perpendicular bisector of AB. It is the locus of points equidistant from A and B, so we can write its equation as

$$\begin{aligned} (x-0)^2 + (y-0)^2 &= (x-b)^2 + (y-0)^2 \\ \implies \quad x^2 + y^2 &= x^2 - 2bx + b^2 + y^2 \\ \implies \quad 0 &= -2bx + b^2 \\ \implies \quad x &= b/2. \end{aligned}$$

Note that this passes through the midpoint $F = (b/2, 0)$ of AB, and is indeed perpendicular to AB (which is the x-axis). [So we did verify that the perpendicular bisector is the locus of points equidistant from A and B.]

Next, let's compute the perpendicular bisector of AC. The equation of line AC is $y = (d/c)x$ or $cy - dx = 0$. The equation of a line perpendicular to it has the form $cx + dy = g$. Plugging in the midpoint $E = (c/2, d/2$ of AC into this equation, we get $g = (c^2 + d^2)/2$.

Therefore, the equation of the perpendicular bisector of AC is

$$cx + dy = \frac{1}{2}(c^2 + d^2).$$

Now, we can take the intersection of these two perpendicular bisectors to figure out the location of the circumcenter O. We just saw the two equations simultenously. The first equation tells us x, and plugging that value into the second, we can figure out $y = (c^2 + d^2 - bc)/(2d)$. In other words, the location of the circumcenter is

$$O = \left(\frac{b}{2}, \frac{c^2 + d^2 - bc}{2d}\right).$$

What about the centroid G? It is $2/3$ of the way from C to the midpoint F of AB. Therefore it can be written as

$$\begin{aligned} G &= \frac{1}{3}C + \frac{2}{3}F \\ &= \frac{1}{3}(c, d) + \frac{2}{3}(b/2, 0) \\ &= \left(\frac{b+c}{3}, \frac{d}{3}\right). \end{aligned}$$