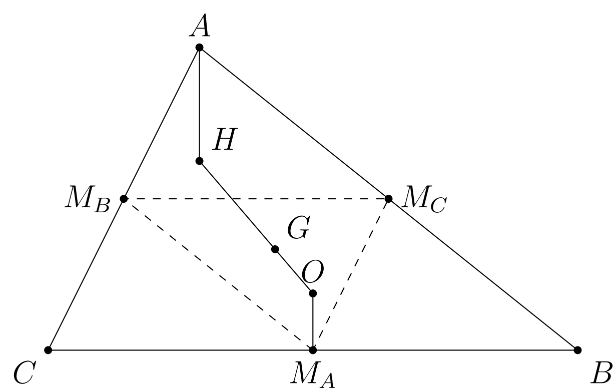
MATH 9B: GEOMETRY CLASSWORK [JAN 11, 2026]
NINE-POINT CIRCLE, EULER LINE.

Theorem 1. *Let O be the circumcenter, H the orthocenter, and G the centroid of a triangle ABC. Then O, G, H like on a straight line, and G divides the segment HO in the ratio $2 : 1$, i.e. $HG = 2GO$.*

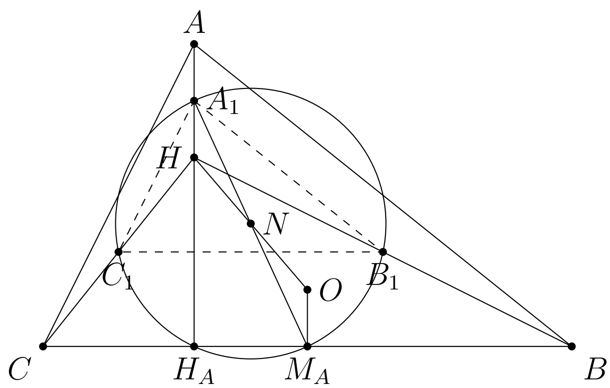


Remark 1. *When $\triangle ABC$ is equilateral, these three points coincide, and the theorem is vacuously true. So we'll assume the triangle is not equilateral. In that case, the line joining HO is called the* **Euler line** *of the triangle.*

Proof. Let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB respectively. Remember that $\triangle M_AM_BM_C$ is similar to $\triangle ABC$ and with similarity ratio $1 : 2$. In fact, since G divides the medians in the ratio $2 : 1$, the two triangles are not only similar but homothetic, with center of homothety G and ratio $2 : -1$. Now, M_AO is the perpendicular bisector of BC, so it is perpendicular to M_BM_C (which is parallel to BC); therefore it is the altitude from M_A in the triangle $M_AM_BM_C$. Similarly for the other sides; therefore O is the orthocenter of $\triangle M_AM_BM_C$. By homothety, H and O are homothetic with G as the center of homothety and ratio $2 : -1$. This proves the theorem.

As a by-product of this proof, notice that $AH = 2OM_A$ etc. □

Theorem 2. *In a triangle ABC, let the midpoints of the opposite edges be M_A, M_B, M_C respectively. Let H_A, H_B, H_C be the feet of the altitudes from A, B, C respectively. Let H be the orthocenter of the triangle, and let A_1, B_1, C_1 be midpoints of HA, HB, HC respectively. These nine points $M_A, M_B, M_C, H_A, H_B, H_C, A_1, B_1, C_1$ are all on a single circle with center N which is the midpoint of HO (where O is the circumcenter). The radius of the nine-point circle is half the circumradius of $\triangle ABC$.*



Proof. Consider a homothety with center H and ratio $1:2$. It contracts A, B, C to A_1, B_1, C_1. Therefore, triangle $A_1B_1C_1$ is homothetic (and similar) to the original $\triangle ABC$ with ratio $1/2$. In particular, the circumcenter O of $\triangle ABC$ goes to the circumcenter of $\triangle A_1B_1C_1$, which must therefore be the midpoint of HO, which we will call N. Let the circumcircle of $A_1B_1C_1$ be called $\mathcal{C}_N$; it has center N. Next, note that reflection (i.e. homothety with ratio -1) about N must take M_A to A_1, since this homothety takes H to O, and since $HA_1 = HA/2 = OM_A$ and HA_1 is parallel to OM_A. Therefore, $NA_1 = NM_A$ which shows M_A lies on $\mathcal{C}_N$. Similarly, M_B and M_C lie on $\mathcal{C}_N$. Next, note that A_1M_A is a diameter of the circle $\mathcal{C}_N$, and since $\angle A_1H_AM_A = 90°$, H_A lies on the circle $\mathcal{C}_N$ as well. Similarly for H_B and H_C. This shows the existence of the nine-point circle. Finally, notice that since the nine-point circle is the circumcircle of $\triangle A_1B_1C_1$, which is homothetic to $\triangle ABC$ with similarity factor $1/2$, its radius is half that of the circumcircle of triangle ABC. This finishes the proof of the theorem. □