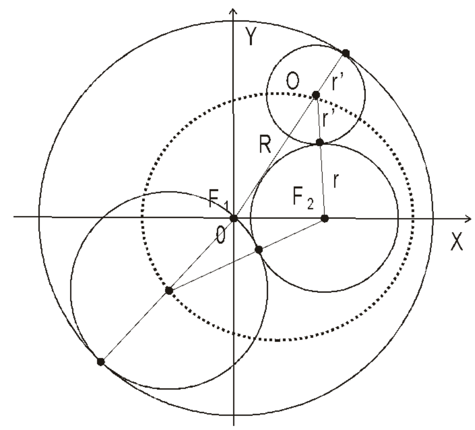
February 22, 2026 Math 9b

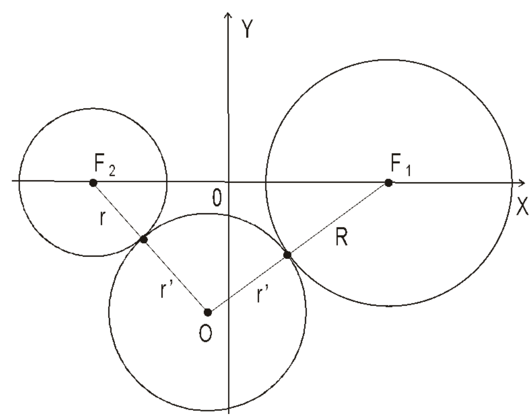
Geometry.

Ellipse. Hyperbola. Parabola (continued).

Alternate definitions of ellipse, hyperbola and parabola: Tangent circles.



Ellipse is the locus of centers of all circles tangent to two given nested circles (F_1, R) and (F_2, r). Its foci are the centers of these given circles, F_1 and F_2, and the major axis equals the sum of the radii of the two circles, $2a = R + r$ (if circles are externally tangential to both given circles, as shown in the figure), or the difference of their radii (if circles contain smaller circle (F_2, r).).



In the figure above, the circle of radius is tangent to the two nested circles. We have $|OF_1| = R - r'$ and $|O F_2| = r + r'$, and therefore $|OF_1| + |O F_2| = R + r = 2a$.

Next, consider circles (F_1, R) and (F_2, r). that are not nested. Then the loci of the centers O of circles externally tangent to these two satisfy $|OF_1| - |O F_2| = R - r$.

Hyperbola is the locus of the centers of circles tangent to two given non-nested circles. Its foci are the centers of these given circles, and the vertex distance $2a$ equals the difference in radii of the two circles.

As a special case, one given circle may be a point located at one focus; since a point may be considered as a circle of zero radius, the other given circle—which is centered on the other focus—must have radius $2a$. This provides a simple technique for constructing a hyperbola.

Exercise. Show that it follows from the above definition that a tangent line to the hyperbola at a point P bisects the angle formed with the two foci, i.e., the angle F_1PF_2. Consequently, the feet of perpendiculars drawn from each focus to such a tangent line lie on a circle of radius a that is centered on the hyperbola's own center.

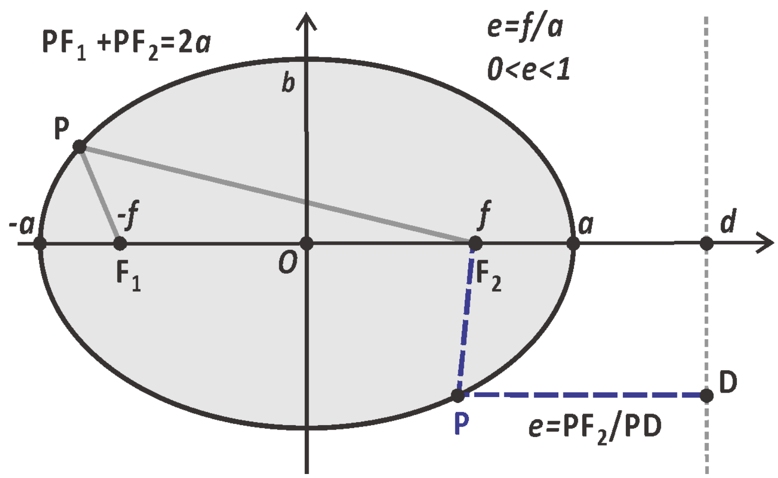
If the radius of one of the given circles is zero, then it shrinks to a point, and if the radius of the other given circle becomes infinitely large, then the "circle" becomes just a straight line.

Parabola is the locus of the centers of circles passing through a given point and tangent to a given line. The point is the focus of the parabola, and the line is the directrix.

Alternate definitions of ellipse, hyperbola and parabola: Directrix and Focus.

Parabola is the locus of points such that the ratio of the distance to a given point (focus) and a given line (directrix) equals 1.

Ellipse can be defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.



Hyperbola can also be defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$. This constant is the eccentricity of the hyperbola. By symmetry a hyperbola has two directrices, which are parallel to the conjugate axis and are between it and the tangent to the hyperbola at a vertex.

In order to show that the above definitions indeed those of an ellipse and a hyperbola, let us obtain relation between the x and y coordinates of a point P (x, y) satisfying the definition. Using axes shown in the Figure, with focus F_2 on the X axis at a distance l from the origin and choosing the Y-axis for the directrix, we have

$$\frac{\sqrt{(x-l)^2+y^2}}{x} = e$$

$$(x - l)^2 + y^2 = (ex)^2$$

$$x^2(1 - e^2) - 2lx + l^2 + y^2 = 0$$

$$(1 - e^2)\left(x^2 - 2x\frac{l}{1-e^2} + \left(\frac{l}{1-e^2}\right)^2\right) + y^2 = \frac{l^2}{1-e^2} - l^2 = \frac{e^2l^2}{1-e^2}$$

Finally, we thus obtain,

$$\frac{(x-\frac{l}{1-e^2})^2}{\frac{e^2l^2}{(1-e^2)^2}} + \frac{y^2}{\frac{e^2l^2}{1-e^2}} = 1$$

Which is the equation of an ellipse for $1 - e^2 > 0$ and of a hyperbola for $1 - e^2 < 0$. In each case the center is at $x = x_0 = \frac{l}{1-e^2}$ and $y = y_0 = 0$, and the semi-axes are $a = \frac{e\,l}{(1-e^2)}$ and $b = \frac{e\,l}{\sqrt{|1-e^2|}}$, which brings the equation to a canonical form,

$$\frac{(x-x_0)^2}{a^2} \pm \frac{(y-y_0)^2}{b^2} = 1$$

We also obtain the following relations between the eccentricity e and the ratio of the semi-axes, a/b: $\frac{b}{a} = \sqrt{|1 - e^2|}$, or, $e = \sqrt{1\pm\left(\frac{b}{a}\right)^2}$, where plus and minus sign correspond to the case of a hyperbola and an ellipse, respectively.

Curves of the second degree.

<u>A curve of the second degree</u> is a set of points whose coordinates in some (and therefore in any) Cartesian coordinate system satisfy a second order equation,

$$a_{11}x^2 + a_{12}xy + a_{22}x^2 + 2b_1x + 2b_2y + c = 0$$