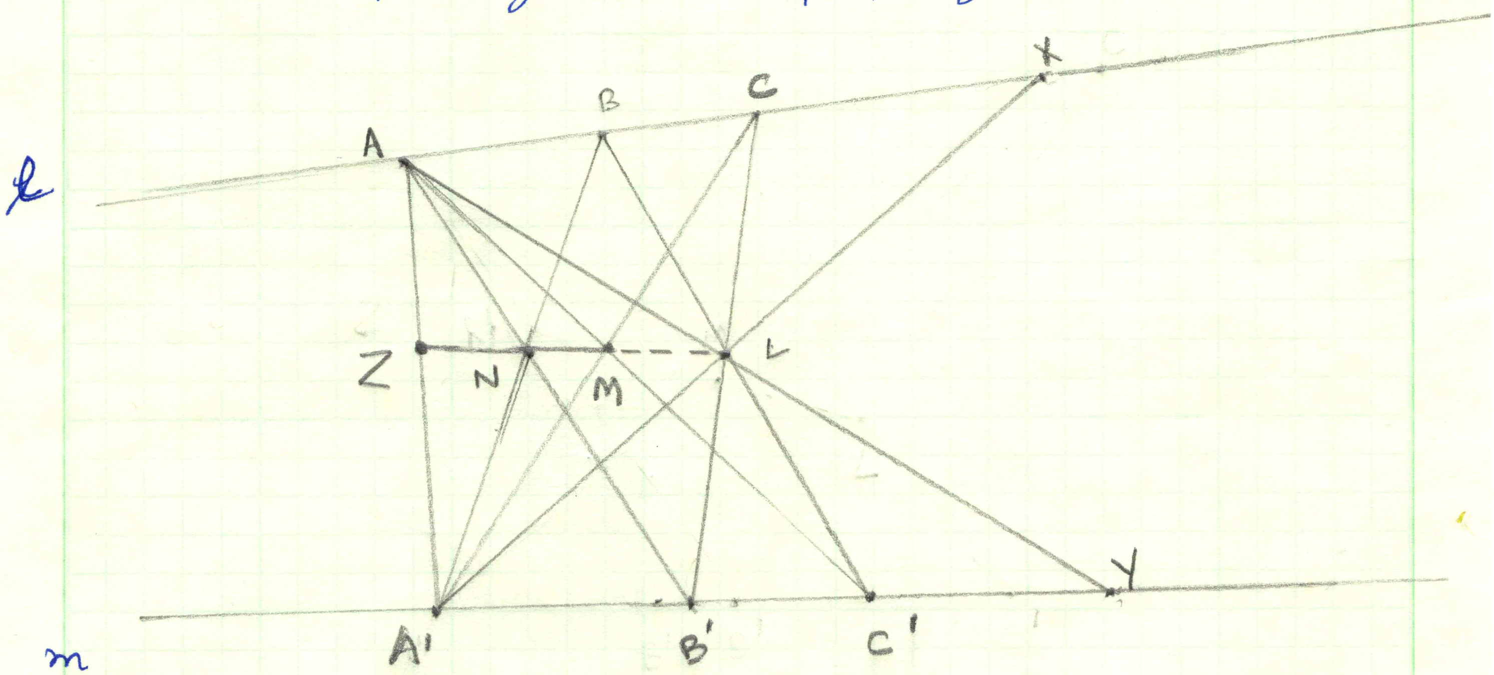
Pappus's Theorem

Let A, B, C be points on a line ℓ, and A', B', C' be points on a line m. Let $L = BC' \cap B'C$, $M = AC' \cap A'C$, $N = AB' \cap A'B$. Then L, M, N are collinear.

Proof: There are some complicated proofs using several applications of Menelaus' theorem. But since we've learned about the cross-ratio, we'll give a simple proof using it.



Extend the line MN and let it intersect AA' in Z.

Let $X = A'L \cap \ell$ and $Y = AL \cap m$.

Now $(AB; C, X) = (Y, C'; B', A')$ (project from L to line m)

$= (A', B'; C', Y)$ (just rearranging the cross-ratio)

But also $(A, B; C, X) = (Z, N; M, *)$ where $* = A'X \cap MN$ (by projection from A' onto MN)

and $(A', B'; C', Y) = (Z, N; M, \dagger)$ where $\dagger = AY \cap MN$

(by projection from A onto MN)

This forces $* = \dagger$. In other words, $A'X$, AY, and MN are all concurrent. But $A'X \cap AY = L$ by definition. So MN passes through L. $\blacksquare$

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