MATH 9B: TRIGONOMETRY [APRIL 26, 2026]
REVIEW OF PROBLEMS, TRIGONOMETRIC SERIES.

Review of some problems

1. Show that
$$\cos^2 \alpha + \cos^2\left(\frac{2\pi}{3} + \alpha\right) + \cos^2\left(\frac{2\pi}{3} - \alpha\right) = \frac{3}{2}.$$
We can use the angle sum formulae to write
$$\cos\alpha = \cos\alpha$$
$$\cos\left(\frac{2\pi}{3} + \alpha\right) = -\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha$$
$$\cos\left(\frac{2\pi}{3} + \alpha\right) = -\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha$$

Square and add these to get our result.

2. Without using a calculator find $\sin 75°$, $\cos(\pi/18)$.

We have
$$\sin 75° = \cos 15° = \cos(30°/2) = \sqrt{\frac{1}{2}(1 + \cos 30°)} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$
Similarly,
$$\cos\frac{\pi}{8} = \cos\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2+\sqrt{2}}{4}}.$$
You can similarly show that
$$\cos\frac{\pi}{2^{n+1}} = \sqrt{\frac{2+\sqrt{2+\sqrt{2+\dots}}}{4}}$$
with n square-root signs in all.

Trigonometric series

Problem 1. Using the trigonometric formulae we have been using, find the sum of the following trigonometric series.
$$S_N = \sin x + \sin 2x + \sin 3x + \cdots + \sin Nx$$

Solution. Multiplying the sum with $2\sin\frac{x}{2}$, we can use
$$\sin x \sin\frac{x}{2} = \cos\frac{x}{2} - \cos\frac{3x}{2}$$
$$\sin 2x \sin\frac{x}{2} = \cos\frac{3x}{2} - \cos\frac{5x}{2}$$
$$\dots$$
$$\sin Nx \sin\frac{x}{2} = \cos\left(Nx - \frac{x}{2}\right) - \cos\left(Nx + \frac{x}{2}\right)$$

Adding these we obtain

$$S_N \cdot \sin\frac{x}{2} = \cos\frac{x}{2} - \cos\left(Nx + \frac{x}{2}\right) = 2\sin\frac{Nx}{2}\sin\frac{(N+1)x}{2}$$

and therefore

$$S_N = \frac{\sin\frac{Nx}{2}\sin\frac{(N+1)x}{2}}{\sin\frac{x}{2}}.$$

More generally, for integers $a < b$, you can calculate the sum

$$S_{a,b} = \sin ax + \sin(a+1)x + \cdots + \sin bx$$

in the same way.

Problem 2. Find the sum of the series.

$$C_N = \cos x + \cos 2x + \cos 3x + \cdots + \cos Nx$$

Solution1: Similar to the previous example, we can multiply by $\sin(x/2)$, and sum to get

$$C_N = \frac{\sin\frac{Nx}{2}\cos\frac{(N+1)x}{2}}{\sin\frac{x}{2}}.$$

Solution2: We can also write

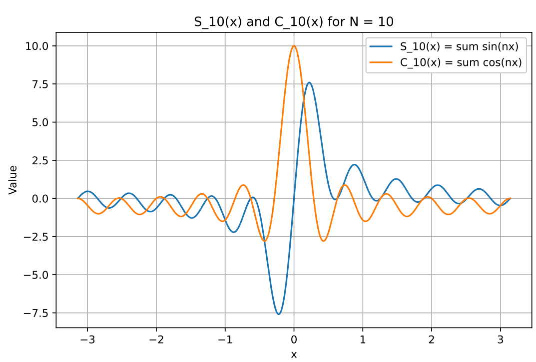
$$C_N = \cos x + \cos 2x + \cos 3x + \cdots + \cos Nx$$
$$C_N = \cos Nx + \cos 2x + \cos 3x + \cdots + \cos x$$

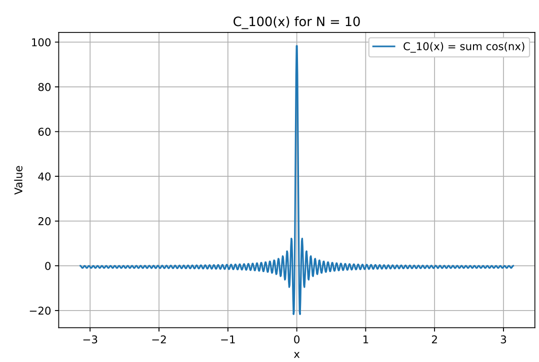
and add to get

$$\begin{aligned} 2C_N &= (\cos x + \cos Nx) + (\cos 2x + \cos(N-1)x) + \cdots + (\cos Nx + \cos x) \\ &= 2\cos\left(\frac{N+1}{2}x\right)\cdot\left(\cos\left(\frac{-(N-1)}{2}x\right) + \cos\left(\frac{-(N-3)}{2}x\right)\cdots + \cos\left(\frac{(N-1)}{2}x\right)\right) \end{aligned}$$

where the sum is of the sort considered in the previous problem, and we solve it as before.

It is interesting to look at the function $C_N(x)$. Intuitively, for $x = 0$ all the terms in the sum are equal to 1, and the sum equals to N, the number of terms. For $x \neq 0$, it consists of many positive and negative terms, which tend to cancel out. Note that we only plotted the function below from $-\pi$ to π, since it is periodic with period 2π.





The study of decomposition of periodic functions as trigonometric sums is part of the subject known as Fourier analysis.