MATH 9B [03/22/2026] - ALGEBRA
COMPLEX NUMBERS

Complex numbers: trigonometric form. Geometric interpretation.

Cartesian and polar forms. Recall from last time that we studied complex numbers $z = x + yi$. This form (x, y) is called the Cartesian form of the complex number z. There is another way to represent the number z as a vector OZ in the complex plane: by specifying its length r, and the angle θ it makes with the x-axis (the "real" axis).

We can go between these as follows: from r and θ we can write

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

so $z = r(\cos\theta + i\sin\theta)$.

For the other way, given x and y, we have

$$r^2 = r^2(\cos^2\theta + \sin^2\theta) = x^2 + y^2 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2}.$$

For θ, note that we can solve for it by specifying

(*) $$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}.$$

(Sometimes this is abbreviated to

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \arctan(y/x).$$

Note: A word of caution is necessary here. The angle θ is only defined up to a multiple of 2π or $360°$. The set of equations (*) is enough to specify θ in this way. But the latter equation, where we only give $\tan\theta$, is not enough (it specifies θ only up to a multiple of π).

Now, let's try to understand what some common operations and equations in the polar form correspond to.

Exercise:

1. If (r, θ) is the polar form of z, what is the polar form of $\bar{z}$?
2. What does the locus $r = 1$ correspond to in the complex plane?
3. What does the locus $\theta = \pi/3$ correspond to in the complex plane?
4. How would you write the line $y = mx$ in polar form?

Consider a complex number $z = r(\cos\theta + i\sin\theta)$, and assume $r > 0$. Then we can write its reciprocal as

$$z^{-1} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}\cdot\frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} = \frac{1}{r}\cdot\frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \frac{1}{r}\cdot(\cos\theta - i\sin\theta)$$

Note that this is the same as $\bar{z}/|z|^2$, which we saw last time.

Multiplication of complex numbers. How does the polar form work with multiplying complex numbers? If we have two complex numbers $z = a + ib$ and $w = c + id$, then we know

$$zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Last time we observed that $|zw|^2 = |z|^2|w|^2$, which gives us the nice identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2,$$

telling us that if we have two numbers m and n which are each a sum of two squares, then so is their product mn. (It is an interesting exercise in number theory to figure out which natural numbers are the sums of two squares.)

Now let's go to the polar form. Say $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. Then we have

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= rs\big(\cos(\theta + \phi) + i\sin(\theta + \phi)\big). \end{aligned}$$

So the modulus of z and w multiply, and the arguments (angles) add, when we take their product.

This is a generalization of the fact we saw last time that if we multiply a complex number by i, the corresponding vector is rotated $90°$ counterclockwise. Here, multiplying z by $w = s(\cos\phi + i\sin\phi)$ does two things: it scales the vector by s (the modulus of w) and it rotates it by ϕ (the argument of w). In particular, if we multiply by a unit complex number $w = \cos\phi + i\sin\phi$, that operation corresponds to just a rotation by ϕ.

De Moivre's formula.

Theorem. For any complex number $z = r(\cos\theta + i\sin\theta)$ and any natural number n, we have

$$z^n = \big(r(\cos\theta + i\sin\theta)\big)^n = r^n(\cos n\theta + i\sin n\theta)$$

Proof. This is basically a proof by induction, but informally the idea is that

$$\begin{aligned} z^n &= r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta) \ldots r(\cos\theta + i\sin\theta) \\ &= r^n(\cos\theta + i\sin\theta) \cdot (\cos\theta + i\sin\theta) \ldots (\cos\theta + i\sin\theta) \\ &= r^n(\cos n\theta + i\sin n\theta) \end{aligned}$$

since there are n factors and the angles add. □

Exercise: Expand $(a + ib)^3$. Now use $a = \cos\theta$ and $b = \sin\theta$ to write $\cos 3\theta$ and $\sin 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise: Check that De Moivre's formula holds for $n = -1$ and therefore also for all negative integer values of n.

How to take n'th roots of complex numbers. De Moivre's theorem, run in reverse, allows us to take n'th roots of complex numbers as well. Suppose we want to solve the equation

$$w^n = z$$

where $z = r(\cos\theta + i\sin\theta)$ is given to us, and n is some natural number. Then we can suppose $w = s(\cos\phi + i\sin\phi)$. Then the equation becomes

$$(\dagger) \qquad s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

From equation (†), we can equate the modulus to get $s^n = r$, so $s = r^{1/n}$. We also need to equate the argument, which is only defined up to 2π. This tells us that

$$n\phi = \theta \pmod{2\pi}$$

which gives us the n possibilities

$$\phi_1 = \frac{\theta}{n}, \phi_2 = \frac{\theta}{n} + \frac{2\pi}{n}, \ldots, \phi_n = \frac{\theta}{n} + \frac{2\pi}{n}(n-1).$$

These n possibilities $\phi_1, \ldots, \phi_n$ for ϕ give n distinct solutions

$$r^{1/n}(\cos\phi_j + i\sin\phi_j)$$

for w satisfying $w^n = z$. Thus, this equation has exactly n roots.

This fact is a special case of the phenomenon that an n'th degree polynomial equation with complex coefficients has exactly n solutions in the complex numbers (counted with multiplicity). This more general theorem is called the *Fundamental Theorem of Algebra*.

Example: Let's solve $w^3 = 1$. By the above calculation, we must have $|w| = 1$, and the possibilities for the argument are $0, 2\pi/3$, and $4\pi/3$. Therefore, recalling the sines and cosines of these angles, the cube roots of unity are

$$1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}.$$

Another way to solve the equation $w^3 - 1 = 0$ is to factor the polynomial as $(w-1)(w^2+w+1)$. Then solve the quadratic in the usual way. The quadratic factor $w^2 + w + 1$ is an example of a *cyclotomic* polynomial.

Example: Calculate the square root of i.

We know that i has modulus 1 and argument $\pi/2$. So one of the square roots of i will have modulus 1 and argument $\pi/4$, and therefore it is

$$\cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

Exercise: Solve the equation $w^4 + w^3 + w^2 + w + 1 = 0$.

Next time, we'll see how to solve equations of the third and fourth degrees. Beyond that, it is generally impossible to write closed-form solutions of equations of degree 5 and beyond, using simple operations such as square roots (or n'th roots). This is a deep fact, whose known proofs require some fairly advanced algebra (Galois theory).