MATH 9B [03/15/2026]
ALGEBRA: NUMBER SETS. RATIONAL AND IRRATIONAL NUMBERS. REAL NUMBERS.

Number sets

Some of the commonly used number sets, such as the set of all digits used in a particular number system, or the set of all permutations of n objects, are finite (have a finite number of elements). Others, like the set of prime numbers, are infinite. Some of the common number sets that we are familiar with are:

1. The set $\{0, 1\}$ of two digits used in the binary number system.
2. The set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of positive integers from 0 to 9 used as the decimal digits.
3. The set $\mathbb{N}$ of natural numbers.
4. The set $\mathbb{Z}$ of integers.
5. The set $\mathbb{Q}$ of rational numbers $\{m/n : m, n \in \mathbb{Z}, n \neq 0\}$.
6. The set $\mathbb{R}$ of all real numbers.
7. The set $D = \mathbb{R} - \mathbb{Q}$ of irrational numbers.

Strictly speaking, we haven't formalized the notion of real numbers, though we have an intuitive notion of what the real line means, and we can also compute with them (as decimal expansions, for instance).

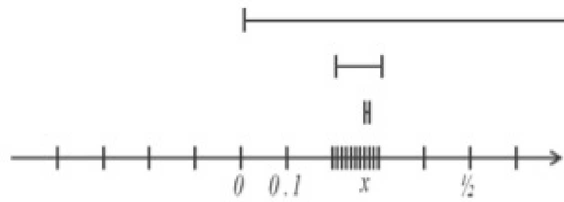
How might we formally define the real numbers. We can try to proceed by defining $\mathbb{N}$, then $\mathbb{Z}$, then $\mathbb{Q}$, and then $\mathbb{R}$.

To any non-empty finite set there is a corresponding natural number, the number of elements in this set. Any two sets that have the same number of elements can be related by a bijection, and all such sets form an equivalence class, corresponding to this natural number. Thus, natural numbers arise as a characteristic of equivalence classes of finite sets having the same number of elements. Georg Cantor, the originator of set theory, in 1874–1884 extended this concept to infinite sets, where equivalence classes will be characterized by "number of elements" in an infinite set, transfinite numbers which are formally infinite.

Once we have defined $\mathbb{N}$ as the set of cardinalities of finite sets, we may define $\mathbb{Z}$ by adding in negatives of elements of the natural numbers, and 0. We can define the usual notions of addition, negation, and multiplication on natural numbers and extend these to integers. Next, we can take quotients by nonzero integers, to obtain the rational numbers. To define the real numbers, we have to do something more interesting. First, let us try to build some intuition by understanding some properties of the rational and irrational numbers. To start with, note that there are clearly some numbers which are irrational, such as $\sqrt{2}$, $\sqrt{3}$, etc. So D is a non-empty set. It is also infinite, since if x is irrational, then $x + q$ is irrational for any $q \in \mathbb{Q}$ (why?)

Density of $\mathbb{Q}$

If we represent rational numbers by points on the number line, we will see that the resulting set of the rational points is dense: there is no interval on the number line, no matter how small, which is free of rational points.



Theorem. Within any interval (A, B), no matter how small, there are rational points.

Proof. This is easily proven by looking at finer and finer meshes (say arithmetic progressions) of rational numbers, as follows. For any n, we can consider all the points $m/10^n$, as m ranges over the integers. When n is really large, we'll see that one of these points must lie inside the given interval (A, B). Since these consecutive points are separated by $1/10^n$, which becomes arbitrarily small as n gets larger, as soon as $10^{-n} < B - A$, one of these endpoints must lie in (A, B) : consider the largest m for which $m/10^n$ is to the left of the interval (A, B). Then $(m+1)/10^n$ is either in the interval or to the right of it. In the latter case, we must have $(m+1)/10^n - m/10^n \geq B - A$, which contradicts our assumption that n is chosen large enough for $10^{-n} < B - A$. So the rational number $(m+1)/10^n$ lies inside (A, B). □

(Note: in the above proof, we could just have used n instead of 10^n.)

Corollary. Any interval, no matter how small, contains infinitely many rational points.

Proof. Indeed, if (A, B) only had finitely many rational points, say $C_1 < \cdots < C_n$, then (C_1, C_2) would have no rational points, contradicting the theorem. □

It would seem from the above that there must be vastly more rational numbers than there are integers: integers are sparse on the number line, there are infinitely many lengths 1 segments devoid of integers! Surprisingly, though, the set of the rational numbers is countable; we showed this by constructing an injective function f from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$, say by writing any $q \in \mathbb{Q}$ as a/b in reduced form, and defining $f(q) = (a, b)$. We already know $\mathbb{Z} \times \mathbb{Z}$ is countable (why?), and so $\mathbb{Q}$ is too.

What about the set of real numbers, which contains both rational and irrational numbers? We saw a proof that it is not countable, by considering real numbers as decimal expansions, and showing that there are uncountably many of these (Cantor's diagonal argument). This approach gives one way to formally define the real numbers. Let us first try to understand some informal properties of decimal expansions.

Real numbers as infinite decimals

Consider decimals between 0 and 1, defined by all possible sequences of digits after the decimal point, $0.a_1a_2a_3 \ldots a_n \ldots$, where each $a_i \in \{0, 1, 2, \ldots, 9\}$. Clearly, all other decimals, lying outside the $[0, 1]$ interval, are obtained by simply adding an integer. Some of these decimals denote rational numbers. In particular, such are the sequences with only a finite number of non-zero digits:

$$\frac{a_1 \ldots a_n}{10^n} = 0.a_1 \ldots a_n = 0.a_1 \ldots a_n(0)$$

where (0) denotes a periodic sequence of zeros with period 1.

Theorem. Any infinite decimal that ends in a periodically repeating sequence of digits, such as

$$0.a_1a_2\dots a_n(x_1x_2\dots x_k) = 0.a_1a_2\dots a_nx_1x_2\dots x_kx_1x_2\dots x_k\dots$$

represents a rational number p/q.

Proof. We calculate by using the decimal expansion

$$\begin{aligned} 0.a_1a_2\dots a_n(x_1x_2\dots x_k) &= \frac{a_1\dots a_n}{10^n} + \frac{x_1x_2\dots x_k}{10^{n+k}} + \frac{x_1x_2\dots x_k}{10^{n+2k}} + \frac{x_1x_2\dots x_k}{10^{n+3k}} + \dots \\ &= \frac{a_1\dots a_n}{10^n} + \frac{x_1x_2\dots x_k}{10^{n+k}}\left(1 + \frac{1}{10^k} + \frac{1}{10^{2k}} + \dots\right) \\ &= \frac{a_1\dots a_n}{10^n} + \frac{x_1x_2\dots x_k}{10^{n+k}} \cdot \left(\frac{1}{1-10^{-k}}\right) \\ &= \frac{a_1\dots a_n}{10^n} + \frac{x_1x_2\dots x_k}{10^n(10^k-1)} \end{aligned}$$

which is a rational number. □

The opposite statement is also true.

Theorem. Rational numbers that are not finite decimals, are periodic infinite decimals.

Proof. A rational number, p/q, is expanded into a decimal fraction by performing long division by an integer q. At each step of this division, there must be a non-zero remainder, otherwise the decimal fraction is finite. However, all possible remainders are integers between 1 and $q-1$, which means that some remainder, r, has to repeat within at most q divisions. After that, the sequence of remainders between the first and the second appearance of r will repeat periodically, thus yielding a periodic decimal fraction. □

Exercise. Show that decimals $0.09999\dots = 0.0(9)$ and 0.1 represent the same (rational) number.

Definition of real numbers

Infinite decimals that are not periodic represent numbers that are not rational, and therefore are irrational numbers. Real numbers are represented by all possible decimal expansions, both finite and infinite. While this definition of real numbers is quite simple and straightforward, it relies on a particular (decimal) number system, and does not offer an equally simple way to port algebraic operations, which have been introduced for rational numbers, to the real numbers.

Real numbers as nested intervals: The construction with a set of nested intervals with rational endpoints illustrated in the figure above, provides a natural way to define the irrational numbers. This definition is based on a geometrical postulate that an infinite set of nested intervals whose length tends to zero (ie is smaller than any arbitrarily chosen small number for all intervals except for a finite subset) has precisely one point common to all intervals. This point, even though it is defined by the nested intervals with rational endpoints, itself can be either rational, or irrational. The set of all such points, determined by all possible sets of nested rational intervals, defines all real numbers.

Real numbers as Dedekind cuts: An alternative, axiomatic way to extend rational numbers and define real numbers was proposed by Richard Dedekind in 1872. Let us assume that we can divide the set of rational numbers $\mathbb{Q}$ into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$:

$$\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b.$$

The partition of the set of rational numbers into two such subsets is called a Dedekind cut. We think of the cut as defining a number which is the "upper boundary" of $\mathbb{Q}_<$, and also the "lower boundary" of $\mathbb{Q}_>$.

There are three possibilities in such a partition:

1. $\mathbb{Q}_>$ contains a smallest element, $b_0 \in \mathbb{Q}_>$, $\forall b \in \mathbb{Q}_>$, $b_0 \leq b$.
2. $\mathbb{Q}_<$ contains a largest element, $a_0 \in \mathbb{Q}_<$, $\forall a \in \mathbb{Q}_<$, $a \leq a_0$.
3. Neither $\mathbb{Q}_>$ contains a smallest element, nor $\mathbb{Q}_<$ contains a largest element.

In the first two cases, we can identify the Dedekind cut with the rational number b_0 or a_0, respectively. In the third case, the Dedekind cut defines an irrational number. This definition agrees with the definition based on shrinking nested intervals, as each set of such nested rational intervals defines a Dedekind cut, as follows. Suppose we have a nested sequence

$$I_1 = (a_1, b_1) \supset I_2 = (a_2, b_2) \supset I_3 = (a_3, b_3) \dots$$

of nested intervals, with size $b_i - a_i$ going to 0. then we can define $\mathbb{Q}_<$ as $\{x \in \mathbb{Q} : x < a_i \text{ for some } i\}$, i.e the set of rational numbers that are smaller than the left endpoint of any of these intervals. For the right subset, we can define $\mathbb{Q}_> = \{x \in \mathbb{Q} : x \geq a_i \text{ for all } i\}$ the set of rational numbers that are larger than the right endpoint of all of these intervals. By definition, $\mathbb{Q}_<$ and $\mathbb{Q}_>$ form a partition of $\mathbb{Q}$, and they satisfy the condition for a Dedekind cut.

Having defined real numbers as nested intervals, or Dedekind cuts, it is easy to see that all the usual arithmetic operations and properties of rational numbers are transposed to real numbers. For the case of nested intervals, this is accomplished by applying an operation to the rational endpoints of the two intervals that define two real numbers, and associating the result with the third set of nested intervals so obtained. Similarly, operations for the Dedekind cuts are defined by reference to rational sets that define the cut.

For example, to define addition of two real numbers r and s, suppose that r is defined by the nested sequence of intervals $I_1 = (a_1, b_1) \supset I_2 \supset I_3 \dots$, and s defined by $J_1 = (c_1, d_1) \supset J_2 \supset J_3 \dots$. Then $r + s$ can be defined by $I_1 + J_1 \supset I_2 + J_2 \dots$, where $I_1 + J_1 = (a_1 + c_1, b_1 + d_1)$ etc. (If I_k has length $\epsilon_k \to 0$ and $J_k \to 0$ has length δ_k, then $I_k + J_k$ has length $\epsilon_k + \delta_k$, which also tends to 0 as $k \to \infty$.)

If instead r corresponds to the Dedekind cut $\mathbb{Q}_{r,<} \bigcup \mathbb{Q}_{r,>}$ and s to $\mathbb{Q}_{s,<} \bigcup \mathbb{Q}_{s,>}$, then $r + s$ could be defined by the Dedekind cut $\mathbb{Q}_< := \mathbb{Q}_{r,<} + \mathbb{Q}_{s,<} = \{x + y : x \in \mathbb{Q}_{r,<}, y \in \mathbb{Q}_{s,<}\}$ and $\mathbb{Q}_> := \mathbb{Q}_{r,>} + \mathbb{Q}_{s,>}$.

Properties of the real numbers

Ordering and comparison.

1. $\forall a, b \in \mathbb{R}$, one and only one of the following relations holds
- $a = b$
- $a < b$
- $a > b$

2. $\forall a < b \in \mathbb{R}$, $\exists c \in \mathbb{R}$, $(c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity: $\forall a, b, c \in \mathbb{R}$, $(a < b) \wedge (b < c) \Rightarrow (a < c)$.
4. Archimedean property: $\forall a, b \in \mathbb{R}$, $a > b > 0$, $\exists n \in \mathbb{N}$, such that $a < nb$.
5. Continuity. Consider a set of nested segments $[a_n, b_n], n \in \mathbb{N}$, $a_n, b_n \in \mathbb{R}$, $a_1 \leq a_2 \leq \ldots a_n \leq b_1 \leq b_2 \leq b_n$. Then, $\exists A$, such that $\forall n$, $A \in [a_n, b_n]$. If $|a_n - b_n| \to 0$, then such a point A is unique.

Addition and subtraction.

1. $\forall a, b \in \mathbb{R}$, $a + b = b + a$.
2. $\forall a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.
3. $\exists 0 \in \mathbb{R}$, such that $\forall a \in \mathbb{R}$, $a + 0 = a$.
4. $\forall a \in \mathbb{R}$, $\exists(-a) \in \mathbb{R}$, such that $a + (-a) = 0$.
5. $\forall a, b \in \mathbb{R}$, $a - b := a + (-b)$.
6. $\forall a, b, c \in \mathbb{R}$, $a < b \Rightarrow a + c < b + c$.

Multiplication and division.

1. $\forall a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$.
2. $\forall a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. $\forall a, b, c \in \mathbb{R}$, $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. $\exists 1 \in \mathbb{R}$, such that $\forall a \in \mathbb{R}$, $a \cdot 1 = a$.
5. $\forall a \in \mathbb{R}$, $a \neq 0 \Rightarrow \exists(1/a) \in \mathbb{R}$, $a \cdot (1/a) = 1$.
6. $\forall a, b \in \mathbb{R}$, $b \neq 0 \Rightarrow a/b = a \cdot (1/b)$.
7. $\forall a, b, c \in \mathbb{R}$, $(c > 0) \wedge (a < b) \Rightarrow (a \cdot c < b \cdot c)$.
8. $\forall a \in \mathbb{R}$, $a \cdot 0 = 0$, $a \cdot (-1) = -a$.