MATH 9B [03/22/2026] - ALGEBRA

Powers and Roots

Integer Powers. For any integer $n, m \in \mathbb{Z}$:

$$a^n \cdot a^m = a^{n+m}, \qquad \frac{a^n}{a^m} = a^n \cdot a^{-m} = a^{n-m}$$

$$(a^n)^m = a^{n \cdot m} = (a^m)^n \quad (\forall n, m \in \mathbb{Z})$$

Algebraic Roots. For any integer $m \in \mathbb{Z}$, natural $n \in \mathbb{N}$, $a, b \in \mathbb{R}_+$, and $c \in \mathbb{R}$:

- $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0)$
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a} \quad (m > 0)$
- $\sqrt[n]{a} = \sqrt[n \cdot m]{a^m} \quad (m > 0)$
- $\sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (a \neq 0 \text{ if } m \leq 0)$
- $\sqrt[m]{(-a)^m} = \begin{cases} a & \text{if } m = 2k \\ -a & \text{if } m = 2k+1 \end{cases}$

Rational Powers. For any integer $p \in \mathbb{Z}$ and natural $q \in \mathbb{N}$:

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p \quad (a \in \mathbb{R}_+, q \in \mathbb{N}, p \in \mathbb{Z})$$

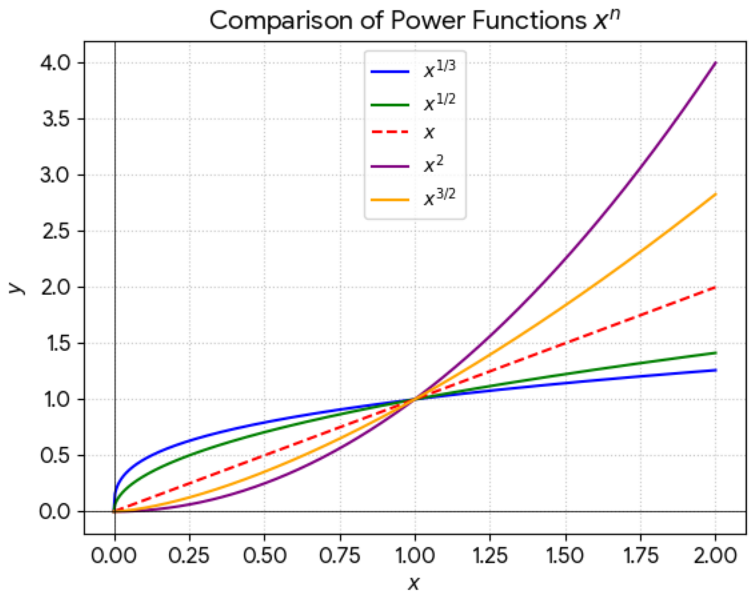
This defines power for rational values of the exponent. The following rules apply in this case, which follow from the the above properties for integer power and roots.

- $(ab)^p = a^p b^p$
- $(\frac{a}{b})^p = \frac{a^p}{b^p}$
- $a^p \cdot a^q = a^{p+q}$
- $(a^p)^q = a^{pq}$
- $(a^p)^{\frac{1}{q}} = a^{\frac{p}{q}}$

Intervals of Monotonic Behavior.

- For $a > 1$, the value of a^p increases when p increases.
- For $0 < a < 1$, the value of a^p decreases when p increases.

For rational $p = m/n$, this can be proven by finding the common denominator of $p = m/n < q = r/s$.



We can extend the definition of powers to irrational numbers x, such as $\sqrt{2}$.

Definition: For an irrational $x \in \mathbb{R}$ and $a > 1$, a^x is a number such that for any rational $p < x$, $a^x > a^p$, while for any rational $p > x$, $a^x < a^p$.

Similarly, for $0 < a < 1$, we can define a^x to be the real number such that for any rational $p < x$, $a^x < a^p$, while for $p > x$, we have $a^x > a^p$. Alternatively, we could have defined a^x in this range to be $1/(1/a)^x$, noting that $1/a > 1$, so we have already defined real powers of it.

To make this definition consistent, we must prove such a number exists and is unique (e.g., via Dedekind section).

Using this definition, we can calculate $2^{\sqrt{2}}$ to any given accuracy by choosing a rational p close enough to $\sqrt{2}$ and computing 2^p. Improving accuracy involves choosing rational numbers closer to $\sqrt{2}$. [Later, we will see other methods, using logarithms and exponentials.]

We can obtain a sequence of rational numbers approaching $\sqrt{2}$ using the continuous fraction:

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \dots}}} \tag{1}$$

Exercise: What are the coefficients a, b, and c here?