MATH 9B [03/22/2026] - ALGEBRA
COMPLEX NUMBERS

Basic definitions

Consider the following way of defining the number $\sqrt{2}$, if we are working in the number system $\mathbb{Q}$ of rationals. We can define a "variable" α, which satisfies the condition $\alpha^2 = 2$. This equation doesn't have a root over the rational numbers (we proved that $\sqrt{2}$ is irrational, by contradiction), and so we "adjoin" a root to extend the number system.

Similarly, an equation which doesn't have any real root is $x^2 + 1 = 0$. (Since if x is a real number, then $x^2 \geq 0$, and so $x^2 + 1 > 0$.)

So we can consider extending the number system $\mathbb{R}$ by adding a symbol i which satisfies the relation:

$$i^2 + 1 = 0.$$

We can think of $\mathbb{C}$ as polynomials in one variable, which we label i rather than x, and which satisfies the extra relation $i^2 = -1$.

Definition. The set $\mathbb{C}$ of complex numbers is the set of numbers of the form $a + bi$, where $a, b \in \mathbb{R}$, with addition and multiplication as usual and with the relation $i^2 = -1$. The number i is called the imaginary unit.

Any real number a can be written as $a + 0i$, hence $\mathbb{R} \subset \mathbb{C}$. For $z = a + bi$, we call a and b the real and imaginary parts (respectively) of z.

$$\mathrm{Re}(z) = a, \quad \mathrm{Im}(z) = b.$$

Numbers of the form bi are called imaginary numbers. This name was coined in the 17th century (as a derogatory term!), as such numbers were regarded by some as fictitious or useless.

Since the operation of addition and multiplication of polynomials satisfy distributive and commutative laws, the same holds for complex numbers.

We can also define the inverse of a nonzero complex number, and therefore we can divide in the complex number system.

Definition. The conjugate of $z = a + bi$ is

$$\overline{z} = a - bi.$$

Theorem. Conjugation satisfies:

$$\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}.$$

Also, the product of a complex number with its conjugate is a non-negative real number.

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Definition. The absolute value of $z = a + bi$ is a non-negative real number $|z|$, defined by

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \quad \Longleftrightarrow \quad |z|^2 = z\overline{z}.$$

For any z, w:

$$|zw| = |z||w|.$$

Definition. The inverse of a non-zero complex number z is z^{-1} such that $zz^{-1} = 1$.

Theorem.

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

Exercise 1. Compute:

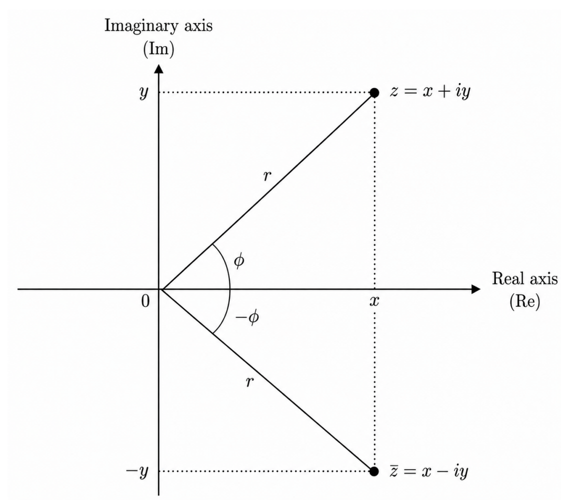
1. $(1+i)^2$
2. $\frac{1}{4+3i}$

Exercise 2. Solve:

1. $z^2 = i$
2. $z^2 + z + 2 = 0$

Geometrical Interpretation of Complex Numbers

There is an obvious one-to-one correspondence between complex numbers $z = x + iy$ and points (x, y) in the plane, as in the figure. In the context of complex numbers, the coordinate plane is called the complex plane (or the Argand plane). The horizontal axis represents real numbers $x \in R$, and the vertical axis represents imaginary numbers iy for $y \in \mathbb{R}$.
In this representation, the absolute value $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ is just the distance of the corresponding point $Z(x, y)$ from the origin O. The following properties of the operations on complex numbers immediately follow:



Properties:

- Addition of two complex numbers $Z_1(x_1, y_1), z_1 = x_1 + iy_1$, corresponds to vector addition : $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$, and $\overrightarrow{OZ_1} + \overrightarrow{OZ_2} = (x_1 + x_2, y_1 + y_2)$.
- Negation of a complex number, i.e. multiplication by -1, corresponds to negation of a vector, i.e. reverses its direction.
- Therefore subtraction of complex numbers corresponds to subtraction of the corresponding vectors.
- Complex conjugation $z \to \bar{z}$, corresponds to reflection about the x-axis.
- Multiplication by a positive real number k, scales the corresponding vector by k.
- Multiplication of a complex number $z = x + iy$ by i changes it to $w = -y + ix$, which correponds to rotating the correspondng vector by $90°$.