MATH 8, FALL 2025
ASSIGNMENT 4: COMBINATIONS WITH REPETITIONS, MULTINOMIAL COEFFICIENTS

Combinations with repetitions

Suppose you have 5 identical marbles and 10 jars, and you have to put each marble in some jar, but no jar can contain two marbles (i.e. "repetitions" or "replacements" are not allowed). The number of ways to do this is the number of ways to choose 5 out of the 10 jars, namely $\binom{10}{5}$.

On the other hand, if you're allowed to put more than one marble in a jar, we reason as follows, via the "stars and bars" diagram:

1. First, place all 5 marbles ("stars") in a row; they are all the same, so the order does not matter.
2. Then, divide this row into $n = 10$ groups by inserting $(10-1) = 9$ dividers ("bars") anywhere in the row; the "bars" are also identical.
3. Counting from the left, each group of stars is the number of marbles for the jar; if two "bars" are next to each other, then the corresponding jar will be empty.

Below is an illustration for $k = 5$ and $n = 10$:

$$\overbrace{\star\star\star\star\star}^{n \text{ stars}} \quad \Longrightarrow \quad \overbrace{|\star|||\star|\star||\star||\star}^{(n+k-1) \text{ stars and bars}}$$

All possible *permutations* of n identical "stars" and $(k-1)$ "bars" give you all possible ways to split into k groups (jars). We counted such permutations in the previous class:

$$(\text{Permutations of } k \text{ stars, } (n-1) \text{ bars}) = \frac{(k+n-1)!}{k!\,(n-1)!} = \binom{k+n-1}{n-1} = \binom{k+n-1}{k}$$

So, in our example, the number of ways to put 5 marbles into 10 jars is

$$(\text{5 marbles into 10 jars}) = \binom{5+10-1}{5} = \frac{14!}{5!\,9!}\,.$$

Further examples:

1. How many ways are there to split 20 passengers in 3 buses? (each bus can seat at least 20).
2. $n = 5$ people play a game of chance consisting of $k = 10$ rounds, and in each round the winner is selected and gets a coin. If all the rounds are alike and the coins are identical, in how many ways can the coins be distributed among the players at the end?
3. How many ways are there to split number 10 into 3 positive integers (order matters)? For example, $10 = 8+1+1,\ 7+2+1,\ 7+1+2\ \ldots$.
4. In a game of 3D chess played in a 8x8x8 cube board, there is a new figure called "squire". It can move only into one of 8 adjacent non-diagonal cubes. If a squire starts from the corner cube and wants to get away from it as far as possible, in how many positions can it end up after 6 moves?

Multinomial coefficient

Another generalization of binomial coefficient $\binom{n}{k}$ is the number of ways to choose, from a collection of n distinct marbles, k_1 to color red (color 1), k_2 to color blue (color 2), ..., k_m to color m, for some number of colors m, where $k_1 + k_2 + \cdots + k_m = n$. (When $m = 2$ and $k = k_1$ and $k_2 = n - k$, we just get $\binom{n}{k}$.) The general answer is called the "multinomial coefficient"

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdots \binom{n-k_1-\cdots-k_{m-1}}{k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}$$

The reason it is called a multinomial coefficient is that there is an analog of the binomial theorem:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} \ldots x_m^{k_m}$$

where $\sum$ stands for summation; the sum is taken over all non-negative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$.

HOMEWORK

1. Calculate how many ways are there
  (a) to distribute 5 identical marbles into 3 different jars;
  (b) to distribute 3 identical marbles into 5 different jars; can you compute it differently?
  (c) stack 10 identical coins in a row of 4 stacks, where each stack must have at least 1 coin?

2. You have a large well-mixed bag of marbles of four different colors, otherwise identical. How many combinations (numbers of marbles of each color) are possible if you randomly pull 4 marbles from a bag? 5 marbles? 6 marbles?

3. How many five-letter "words" are there with the letters in alphabetical order? (Here a "word" is any sequence of letters from the alphabet, "a" through "z".)

There are two versions of this problem: one where all letters are different and the other where letters are allowed to repeat. Try doing both.

4. Imagine that you have a regular cubic dice with numbers 1 to 6 on the sides. You roll the dice a few times and write the combination down. If the order does not matter, how many different combinations can you get when rolling 3 times? 5 times? 10 times?

5. A monomial is a product of powers of variables, i.e. an expression like x^3y^7.
  (a) How many monomials in variables x, y of total degree of exactly 10 are there? (Note: this includes monomials which only use one of the letters, e.g. x^{10}.)
  (b) Same question about monomials in variables x, y, z.
  (c) What about 4-th degree monomials in variables x, y, z, t?
  (d) How many monomials in variables x, y of degree at most 15 are there?

6. Each student in a class of 20 has to pick one day of the week (Monday through Friday) to make a 10-minute presentation. Depending on their choices, the teacher has to make the schedule reserving a fraction of each day to the presentations. How many different schedules are possible (without regard to who and when are making the presentations)?

7. If you have n lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form?

8. A teacher wants to divide her class of 19 students into four groups of size $5, 5, 5, 4$ to work on problems. In how many ways can she do this?

9. Let us put the trinomial coefficients $\binom{n}{a,b,c}$, where $n = a+b+c$, at points in 3d space, putting each $\binom{n}{a,b,c}$ at point (a, b, c). They will fill the first "octant" (3d analog of quadrant in plane) — draw it. By rotating the picture, we get a pyramid, where n^{th} level represents all trinomial coefficients with $a + b + c = n$; this pyramid is the natural analog of Pascal triangle.

Is there a formula expressing the numbers in each next level in terms of numbers in the previous level, as we had for Pascal triangle? What is the sum of all the trinomial coefficients in the n'th level?